Chapter 17
Commentary by Ray Brown and Morris Hirsch

Stretching and Folding in the KIII Neurodynamical Model

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1   **Abstract** In this chapter we provide an alternate view of the KIII model derived from
2 the laws of complexity (Stretching and Folding) rather than the laws of physics. This
3 approach requires the use of Infinitesimal Diffeomorphisms (ID) in place of Ordinary
4 Differential Equations. We indicate how IDs originate and then use them to replicate
5 several examples from the work of Freeman and Kozma. By viewing the KIII theory
6 as a purely mathematical system we anticipate that the KIII Theory will be made more
7 accessible to researchers and scientists unfamiliar with the details of neuroscience
8 and thus offer advances to the KIII Theory from other perspectives.

17.1 Introduction

10   Freeman and Kozma have introduced a paradigm shift in the analysis of neurody-
11 namics by focusing on the mesoscopic structures external to the neurons referred to
12 as the neuropil [1], rather than the dynamics of the neuronal mass only. An under-
13 standing of the amorphous nature of the neuropil, more analogous to a stiff fluid or
14 a shag rug, suggested an entirely new approach to neurodynamical modeling that
15 uses a field or wave paradigm as the means of communication, and the neuropil as
16 the medium over which these waves must travel to relevant regions of the brain. It
17 is on this fundamental wave-based neuropil approach that the KIII model is built.
18 Importantly, their "wave" approach enables an explanation of how intentionality is
19 communicated from the limbic system (the seat of intentionally) to specific regions
20 of the brain in a manner that causes those regions to arrange or configure themselves
21 to perform a desired new task. For example, the wave approach more efficiently

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22 explains how one learns to hit a tennis ball for the first time given that they have
23 never picked up a tennis racquet. This is because the range of dynamical activities
24 involved in learning a new complex task must be performed in a few seconds and
25 involves thought, action and emotion for the first time crossing the entire spectrum of
26 human capability. Since mitosis is not operable within most of the neural mass (and
27 certainly not rapidly), the problem of communicating intent and rapidly learning a
28 new task driven by intent must have a dynamic that relies on speed. A wave dynamic
29 is able to satisfy this specification.

AQ1

30   In this volume, Vitiello describes two alternative approaches to advancing this
31 theory based on the laws of physics. However, addressing how the dynamics, even
32 at mesoscopic level, are transformed into everyday macroscopic behavior driven by
33 human intentionality poses a significant problem which, as yet, has no solution. A
34 fundamental road block is that the dynamics of everyday life cannot be formulated
35 within the framework of the laws of physics. Thus we are stuck with the problem
36 of bridging a dynamical system based on the laws of physics with a dynamical sys-
37 tem most commonly described by statistics. In this chapter we introduce an entirely
38 new approach to neuronal dynamics that side-steps the laws of Newton, physics and
39 statistics. The approach presented here makes a more direct connection between the
40 dynamics of the brain and the dynamics of humans at work and play by formulat-
41 ing both system within the same set of laws, the laws of complexity. The "laws"
42 of complexity are found in the stretching and folding horseshoe paradigm of Smale
43 [2]. This approach has been introduced in the analysis and simplification of physical
44 systems by Hénon [3] in deriving his simplification of the dynamics of the Lorenz
45 system (known now as the Hénon map); but a more extensive analysis is needed to
46 apply this approach to both human and brain dynamics with equal legitimacy. An
47 initial exploration of this concept is found in the Hirsch Conjecture [4] where it is
48 noted that natural systems combine stretching and folding in very small increments
49 as seen in ODEs having chaotic solutions. To obtain a general mathematical expres-
50 sion of complexity dynamics the concept of infinitesimal stretching and folding is
51 introduced in [5]. To bridge the gap between neurodynamics and human dynamics
52 at the macroscopic level it is noted that [5] the laws of complexity apply equally well
53 to human dynamics and brain dynamics.

AQ2

54   Vitielo also mentions another serious problem in understanding brain dynamics:
55 The change in conductance of a single neuron cannot affect the dynamics of the
56 mass action occurring inside the brain. To take this one step further, even minor
57 changes at the mesoscopic level must not affect mass action dynamics. This aspect
58 of neurodynamics is explained by two phenomena. (1) Any determination of the
59 health of a human EEG is based on the morphological properties of the EEG, not its
60 exact time series [6]. (2) The phenomena of sensitive dependence on initial conditions
61 can only be understood in terms of the morphology of the time series in that two
62 nearby trajectories may diverge or become uncorrelated, but their morphology does
63 not change thus the mass action dynamics also do not change with small perturbations
64 at the mesoscopic level [6].

65   In this chapter we shall refer to the KIII model as a bottoms-up approach. Using a
66 bottoms-up approach, the laws of physics are applied to derive a system of differential
67 equations known as the KIII model, Eq. 17.46. We will reverse this approach and use

their work as a springboard to develop a top-down model that uses the "laws of com-
plexity", with which to derive their theory. It is hoped that the combined bottom-up
physics approach and the top-down complexity approach will merge to produce an
even greater methodology with which to analyze and perhaps prove theories of neuro-
dynamics; and, through using the top-down approach, we hope to make research of the
KIII theory available to a wide range of scientists and mathematicians who do not have
an extensive background in neurodynamics, thermodynamics and quantum theory.

17.2 Stretching and Folding Provide an Alternative Approach to the Laws of Physics for Modeling Dynamics

The Newtonian approach to understanding and formulating equations of dynamics
are expressed in his second law: $F = ma$. This formulation is excellent for physics
but obscures the sources of complexity and chaos that can arise in dynamical systems
generally. Hirsch in 1985 [7] set the stage for reexamination of the laws of Newton
with this statement:

> A major challenge to mathematicians is to determine which dynamical systems are chaotic
> and which are not. Ideally one should be able to tell from the form of the differential equation
> (Morris W. Hirsch 1985 [7]).

In [5], following up on the conjecture of Hirsch, it is noted that by rearranging
how the equations of dynamics are written (or simply viewing them from a different
perspective), the form might be able to reveal the presence of chaos where the Newton
approach of $F = ma$ does not. The key to doing this was the observation by Smale
used in guiding the proof of the Smale Birkhoff Theorem [2] that the source of
complexity arose by dividing the Newtonian forces into those that stretch and those
that fold.

As an example, consider the Duffing/Ueda equation without damping:

$$\ddot{y} + y^3 = \beta \cos(t) \tag{17.1}$$

Written in Newtonian form we have

$$\ddot{y} = -y^3 + \beta \cos(t) \tag{17.2}$$

In this form, we are not led to sort out the source of complexity. Now we use paren-
thesis to group the terms as follows:

$$\ddot{y} = (-y^3 \text{ stretcihing}) + (\beta \cos(t) \text{ folding}) \tag{17.3}$$

Recognizing by definition of stretching and folding from [5] we see that $(\beta \cos(t))$
as the folding term and y^3 is the stretching term. We now know that the solution of
this equation must be able to generate complexity not because $F = ma$, but because
the forces involved are stretching and folding. If we apply this approach to the KIII

104 model then we must arrange the differential equations into stretching and folding
105 terms. The trick is to figure out how this translates into diffeomorphisms that contain
106 all the complexity of the KIII model. The thermodynamical KIII model derives ODEs
107 from the known dynamics of fluids moving across membranes; on the other hand, the
108 ID model must begin with an identification of the stretching and folding components.
109 From [4] we know that any diffeomorphism of the form

$$\mathbf{X} \to F(\mathbf{X}) \tag{17.4}$$

111 where $\nabla((\nabla \cdot F)(\mathbf{X})) \neq 0$ is a stretching component; and any diffeomorphism of
112 the form

$$\mathbf{X} \to \exp(A) \cdot \mathbf{X} \tag{17.5}$$

114 where A is a $n \times n$ matrix of constants is a folding component.

115   There is one more step in the derivation. If we numerically integrate an ODE
116 we must break down the numerical solution into discrete, but very short, steps. This
117 implies that we must formulate stretching and fold in small steps, or "infinitesimal"
118 increments. Using the concept of stretching and folding in small increments leads
119 to the concept of Infinitesimal Diffeomorphisms as presented in [4]. By using infin-
120 itesimal steps we blend the dynamics of the two forces nearly continuously as often
121 occurs in the natural world.

122   Now we must get an insight into how to transform an ODE into stretching and
123 folding. There are two steps: (1) Recognizing the stretching and folding components
124 in the ODE of interest; (2) Deriving how stretching and folding appear in ID by
125 converting an ODE into an integral equation. To address (1) we use Eq. 3 from [1]:

$$\ddot{y}_1 + \alpha \dot{y}_1 + \beta y_1 = \beta w_{ee} Q(y_2) \tag{17.6}$$

$$\ddot{y}_2 + \alpha \dot{y}_2 + \beta y_2 = \beta w_{ee} Q(y_1) \tag{17.7}$$

128 Rearranging the equations into stretching and folding we have:

$$\ddot{y}_1 = -(\alpha \dot{y}_1 + \beta y_1 - \text{folding}) + (\beta w_{ee} Q(y_2) - \text{stretching}) \tag{17.8}$$

$$\ddot{y}_2 = -(\alpha \dot{y}_2 + \beta y_2 - \text{folding}) + (\beta w_{ee} Q(y_1) - \text{stretching}) \tag{17.9}$$

131   To address step (2), converting to an integral equation, we provide an intuitive
132 derivation in the next section. Once the basic ideas are fixed, we may jump from
133 ODEs to IDEs with a measure of ease. But first, we must emphasize another aspect
134 of the ID formulation and its relationship to the KIII discrete dynamics.

135   The reformulation of the Freeman and Kozma KIII model as IDEs offers a
136 serendipitous benefit that is related to the discrete dynamics of KIII. The IDE parame-
137 ter h can vary from very small to quite large as shown in [5, 8]. This variation allow
138 us to observe how the dynamics of the brain change with the degree of stretching and
139 folding which, in turn, will be determined by external forces and intentionality. The
140 Hénon study [8] and the studies in [5] show just how dramatic changes occur due to

141   variations in the degree of stretching and folding. For example, in [8] the variation
142   of the IDE parameter, h, can move the neurodynamics through a conventional period
143   doubling process to chaos. One further aspect of the IDE formulation is that it allows
144   us to the study how asymmetry in stretching and folding affects the neurodynamics.
145   In this point-of-view there may be separate parameters, h_s, h_f, for stretching and
146   folding that are driven by external and internal factors. The best example of this is
147   found in [9] where the stretching dynamic is chosen to be a Bernoulli, or Anosov,
148   map and the folding dynamic is an almost periodic map. In [9] they are combined
149   as a weighted sum to demonstrate how remarkably the dynamics can vary as the
150   weighting parameter is moved from 1 (only Bernoulli stretching) to 0 (almost peri-
151   odic folding). Note that, in [9], the Bernoulli component can be further divided into a
152   pure stretching and a pure folding component since Bernoulli is itself a consequence
153   of stretching and folding.

17.3 Infinitesimal Diffeomorphisms First Originated from Integral Equations

156   Two theorems will serve to set the stage of the use of IDEs in biological systems
157   generally. In [6] the ODE

$$\dot{x} + V(x) \cdot x = 0 \quad x(0) = x_0 \tag{17.10}$$

159   was introduced. An integral equation version of this ODE is given by

$$x(t) = \exp\left(-\int_{t_0}^{t} V(x(s))ds\right) \cdot x(t_0) \tag{17.11}$$

161   In higher dimensions $V(x)$ is a square matrix.

162   The importance of the form of this ODE is that it provides an entrance through
163   which to understand the mathematical realization of stretching and folding. To better
164   understand the ideas to come we will use a simplified version of Eq. 17.10:

$$\dot{x} + x = 0 \quad x(0) = x_0 \tag{17.12}$$

166   The solution is obviously $x(t) = \exp(-t) \cdot x_0$. another way to view this solution is
167   to set $t = h$, where h is a small step size.

$$x_{n+1} = \exp(-h) \cdot x_n \quad x_0 \text{ specified} \tag{17.13}$$

169   Now consider

$$\dot{x} + x = f(t) \quad x(0) = x_0 \tag{17.14}$$

171   This is converted into an integral equation by introducing an integrating factor $\exp(t)$

$$x(t) = \exp(-t)\, x_0 + \exp(-t) \int_0^t \exp(s)\, f(s) ds \qquad (17.15)$$

$$x(t) = \exp(-t)\, x_0 + \exp(-t) \int_0^t f(s) d \exp(s) \qquad (17.16)$$

Applying the mean value theorem to Eq. 17.16 we have may obtain a simple iteration scheme that will be presented in Theorem 17.2 to follow.

Let $x(t)$ be a real valued function of a real variable t. We have the following theorem concerning IDEs:

Theorem 17.1 *Assume*

$$\dot{x} + V(x) \cdot x = 0 \quad x(0) = x_0 \qquad (17.17)$$

has a unique bounded solution for every initial condition and that $\|x(t)\| \leq M$ for all $t \in \mathbf{R}$. Let $t_n = n\,h$, for $h \in (0, a]$, for $a < 1$. also, define $\tilde{x}$ as

$$\tilde{x}(t_{n+1}) = \exp(-V(x(t_n))(h)) \cdot x(t_n) \qquad (17.18)$$

then

$$\|x(t_n) - \tilde{x}(t_n)\| \leq K \cdot h \qquad (17.19)$$

for fixed K and all integers n.

Proof A formal proof is deferred to [10]. We sketch some key steps to make the approach clear. The two primary steps are (1) to use mathematical induction to prove the approximation at the nth (the result is clearly true for $n = 0$); and, (2) to use mean value theorems to eliminate integrals in favor of algebraic terms.

Let $\Delta = \|x(t_n) - \tilde{x}(t_n)\|$ then the error at the nth step is given by

$$\Delta = \|(\exp(-V(x(\xi)) \cdot h) - \exp(-V(x(t_0)) \cdot h))) \cdot x(t_0)\| \qquad (17.20)$$

and so

$$\Delta \leq \|\exp(-V(x(\xi)) \cdot h) - \exp(-V(x(t_0)) \cdot h)\| \cdot M \qquad (17.21)$$

and

$$\Delta \leq \|\exp(-V(x(\rho))) \cdot h)\| \cdot \|V(x(t_0)) - V(x(\xi))\| \cdot h) \cdot M \qquad (17.22)$$

Let

$$K_1 = \max_x \|V(x(t))\| \quad \text{and} \quad K_2 = \max_x \|V'(x(t))\|$$

to get

$$\Delta \leq \|\exp(K_1 \cdot h)\| \cdot K_2 \|x(t_0) - x(\xi)\| \cdot h \cdot M \qquad (17.23)$$

$$\Delta \leq \| \exp(K_1 \cdot h)\| \cdot K_2 \|V(x(t)) \cdot x(t) \cdot h\| \cdot h \cdot M \tag{17.24}$$

$$\Delta \leq \| \exp(K_1 \cdot h)\| \cdot K_2 \cdot K_1 \cdot h^2 \cdot M^2 \tag{17.25}$$

showing that the error can be made arbitrarily small. End of sketch.

In particular we have the iteration

$$x_{n+1} = \exp(-V(x_n) \cdot h)) \cdot x_n \tag{17.26}$$

as an approximation to the solution of Eq. 17.10 when all smoothness and bounded-ness assumptions are satisfied.

Now consider Eq. 17.27 with the same assumptions as Theorem 17.1. We roughly sketch the derivation of the relevant IDE and state the theorem afterwards.

$$\dot{x} + V(x) \cdot x = f(t) \quad x(0) = x_0 \tag{17.27}$$

By taking $f(t)$ to be a constant, b, over a very small interval $[t_n, t_{n+1}]$ we may obtain a integral equation containing a convolution. Note that the assumption on f implies that the derivative of f is not too troublesome, i.e., $\|f'(t)\|$ is uniformly bounded over the entire real line.

We need the substitution

$$\frac{d\,W(t)}{dt} = V(x(t))$$

Introducing an integrating factor into Eq. 17.27 and collecting terms we have

$$\frac{d(x(t) \exp(W(t))}{dt} = f(t) \exp(W(t)) \tag{17.28}$$

Integrating over a small interval $[t_n, t_{n+1}]$

$$x(t_{n+1}) \exp(W(t_{n+1})) = x(t_n) \exp(W(t_n)) + \int_{t_n}^{t_{n+1}} f(s) \exp(W(s))ds \tag{17.29}$$

$$x(t_{n+1}) = x(t_n) \cdot \exp(-V(x_n) h) + \exp(-W(t)) \int_{t_n}^{t_{n+1}} f(s) \exp(W(s))ds \tag{17.30}$$

Since we have the solution for the homogeneous equation we only need to consider approximating the inhomogeneous part

$$\exp(-W(t_n)) \int_{t_n}^{t_{n+1}} f(s) \exp(W(s))ds \approx \exp(-W(t_n))\, b \int_{t_n}^{t_{n+1}} \exp(W(s))ds$$
$$\tag{17.31}$$

225 where we have substituted b for $f(t)$ over the interval $[t_n, t_{n+1}]$. At this point we
226 assume $W(t) = t$ to shorten and simplify the discussion so that it better applies to
227 the following presentation. then the integral can be explicitly evaluated ans we arrive
228 a the form of the IDE that will be present in the following discussion.

$$
229 \quad \exp(-(t_n))\, b \int_{t_n}^{t_{n+1}} \exp(s)ds = \exp(-(t_n))\, b\, (\exp(t_{n+1}) - \exp(t_n)) \quad (17.32)
$$

230 This gives us $b\,(\exp(h) - 1)$ for Eq. 17.32. Collecting terms we have the theorem for
231 the case where $V(x) = \alpha$.

232 **Theorem 17.2** *The IDE for Eq. 17.27 is given by*

$$
233 \quad x_{n+1} = \exp(\alpha \cdot h)(x_n - f(n\,h)) + f(n\,h)
$$

234 For $V(x)$ not constant, the derivation is more involved and can be found in [10].

17.4 Deriving IDEs for the KIII Model

236 Note that all equations of the KIII model may be represented in the general form:

$$
237 \quad \frac{d\mathbf{X}}{dt} = \mathbf{A}\mathbf{X} + F(\mathbf{X}, t) \quad (17.33)
$$

238 The origin of IDs comes from converting Eq. 17.33 to an integral equation and
239 then simplifying. An intuitive derivation goes as follows:

$$
240 \quad \exp(-A \cdot (t + h))X(t + h) = \exp(-A \cdot t)X(t)
$$

$$
241 \quad + \int_{t}^{t+h} \exp(-As)F(X(s))ds \quad (17.34)
$$

$$
242 \quad X(t + h) = \exp(A \cdot h)X(t) + \exp(A \cdot (t + h))
$$

$$
243 \quad \int_{t}^{t+h} \exp(-As)F(X(s))ds
$$

$$
244 \quad = \exp(A \cdot h)X(t) + \exp(A \cdot (t + h))
$$

$$
245 \quad \cdot \int_{t}^{t+h} (-A^{-1})F(X(s))d\exp(-As)
$$

$$
246 \quad = \exp(A \cdot h)X(t) + \exp(A \cdot (t + h))((-A^{-1})F(X(\xi))
$$

$$
247 \quad \cdot (\exp(-A(t + h) - \exp(At))
$$

$$
248 \quad = \exp(A \cdot h)X(t) + (-A^{-1})F(X(\xi)(1 - \exp(A \cdot h))
$$

$$
(17.35)
$$

$$X(t + h) = \exp(A \cdot h)(X(t) + (A^{-1})F(X(\xi))(-A^{-1})F(X(\xi)) \tag{17.36}$$

$$X(t_{n+1}) \approx \exp(A \cdot h)(X(t_n) - G(X(t_n)) + G(X(t_n)) \tag{17.37}$$

where $G(X(t)) = (-A^{-1})F(X(\xi))$ and $exp = \exp(A \cdot h)$. This requires that A^{-1} exists. When the solution is an attractor, and F is bounded, the ID provides a very good approximation to the solution of a nonlinear autonomous ODE.

Using this form of the ID justifies looking for solutions to any equation of the form Eq. 17.33 by assuming it has the form of an IDE. The correspondence is this:

$$\dot{\mathbf{X}} = \mathbf{A}\mathbf{X} + F(\mathbf{X}, t) \tag{17.38}$$

$$\mathbf{T}_h(\mathbf{X}) = \exp(\mathbf{B} \cdot h)(\mathbf{X} - G(\mathbf{X}, f(h))) + G(\mathbf{X}, f(h)) \tag{17.39}$$

$$\mathbf{X}_{n+1} = \exp(\mathbf{B} \cdot h)(\mathbf{X}_n - G(\mathbf{X}_n, n \cdot h))) + G(\mathbf{X}_n, n \cdot h) \tag{17.40}$$

This derivation is partly formal, partly experimental. In general, we start with the form of an ID if we are working with an equation of the form of Eq. 17.33 and then we use formal data to obtain the best approximation to the stretching terms and folding terms separately. The folding terms will be captured in the eigenvalues of $\mathbf{B}$ and the stretching terms will be determined by the form of G and its "stretching" parameters. The model derived by shifting our emphasis from KIII ODEs to KIII ID will be referred to as KIII-ID.

From an engineering point-of-view, since we are starting with the known form of the solution, using the KIII-ID the numerical approximation and modeling should be achieved with a significant reduction in computational effort. This may come in the form of a reduction in the number of equations needed to model neurodynamics. We justify abandoning the derivation of a specific time series related to the physics described by the ODEs by the morphology principle of the EEG. This recognizes that it is the "form" of the equations that will best capture neurodynamics rather than an analysis of the physics of fluids or quantum theory.

17.4.1 The Linear ID Provides Fundamental Insights into the Dynamics of Stretching and Folding Systems

A linear Infinitesimal Diffeomorphism (ID) is

$$\mathbf{T}_h(\mathbf{X}) = \exp(\mathbf{A} \cdot h)(\mathbf{X} - F(\mathbf{X})) + F(\mathbf{X}) \tag{17.41}$$

where $\mathbf{A} \neq 0$ is an n by n matrix of constants, $\mathbf{X}$ is an n-vector and $\nabla(\nabla \bullet F) \neq 0$, where F is twice differentiable function on $\mathbf{R}^n$ and $h \neq 0$.

The condition, $\nabla(\nabla \bullet F) \neq 0$ is the definition of stretching.

and when the matrix **A** has some eigenvalues less than 1, this approximation can be extremely accurate, see [12], and the step size h can be as large as 0.5 while retaining the morphological properties of the exact solution, [4]. Note that in Eq. 17.48 the time variable may be absorbed to make the equation *autonomous* by increasing the dimension by 1.

Equation 17.48 is required to have sufficiently smooth derivatives. We rewrite Eq. 17.48 in the form of a transformation on a manifold:

$$\mathbf{T}_h(\mathbf{X}) = \exp(\mathbf{A} \cdot h)(\mathbf{X}_n - F(\mathbf{X}_n)) + F(\mathbf{X}_n) \tag{17.49}$$

The ID, Eq. 17.49, has broad applicability and occurs in a wide range of problems of physics, fluid flow and electronic circuits [13].

More generally, an Infinitesimal Diffeomorphism (ID) is a one-parameter family of maps on $\mathbf{R}^n$ of the form (17.50) where F is a twice differentiable mapping from $\mathbf{R}^n$ to $\mathbf{R}^n$, $G(X)$ is a twice differentiable matrix function of $X \in \mathbf{R}^n$ and $h \neq 0$ is a real parameter.

$$\mathbf{T}_h(\mathbf{X}) = \exp(G(\mathbf{X}) \cdot h)(\mathbf{X} - F(\mathbf{X})) + F(\mathbf{X}) \tag{17.50}$$

As noted previously, the significance of IDs is that they are diffeomorphisms that also have the characteristics of a time series. This fact makes it possible to analyze very complex nonlinear processes more efficiently than by using conventional numerical methods. In addition to the ability to analyze fundamental dynamics, the ID provides an avenue for compression of high-dimensional systems of ODEs due to its similarity to Gaussian integration for second-order ODEs. IDs are particularly well suited to analyze the morphology of nonlinear ODEs of the form (17.46) which includes such equations as the Chua double scroll, the Lorenz system, the Rössler system and the K-neurodynamical models that will be discussed in this paper.

17.5 The Application of IDs to K-Neurodynamics May Result in Useful Simplifications of the ODEs Use to Describe the KIII System

In this section will apply IDs to formulate the K-neurodynamical models. These models will be designated as the K-ID models.

The K0-ID infinitesimal diffeomorphism is a direct translation of the K0 Eq. 1 model [1]. As noted earlier, this translations replaces a single second order ODE with a pair of IDs.

$$\mathbf{X}_n = ((x_n - F(x_n, y_n)); \quad \mathbf{Y}_n = ((y_n - F(x_n, y_n)) \tag{17.51}$$

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} \exp(\alpha \cdot h) \cdot \mathbf{X}_n \cdot \cos(\omega \cdot h) + \mathbf{Y}_n \cdot \sin(\omega \cdot h)) + F(x_n, y_n) \\ \exp(\alpha \cdot h) \cdot \mathbf{Y}_n \cdot \cos(\omega \cdot h) - \mathbf{X}_n \cdot \sin(\omega \cdot h)) + F(x_n, y_n) \end{pmatrix} \tag{17.52}$$

The KI-ID infinitesimal diffeomorphism is a modified version of the KI model, Eq. 3 model in [1] Some abbreviations are needed here:

$$\mathbf{X}_n = ((x_n - F(z_n, w_n)); \ \mathbf{Y}_n = ((y_n - F(z_n, w_n)) \tag{17.53}$$

$$\mathbf{Z}_n = ((z_n - F(x_n, y_n)); \ \mathbf{W}_n = ((w_n - F(x_n, y_n)) \tag{17.54}$$

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \\ w_{n+1} \end{pmatrix} = \begin{pmatrix} \exp(\alpha \cdot h) \cdot \mathbf{X}_n \cdot \cos(\omega \cdot h) + \mathbf{Y}_n \cdot \sin(\omega \cdot h)) + F(z_n, w_n) \\ \exp(\alpha \cdot h) \cdot \mathbf{Y}_n \cdot \cos(\omega \cdot h) - \mathbf{X}_n \cdot \sin(\omega \cdot h)) + F(z_n, w_n) \\ \exp(\alpha \cdot h) \cdot \mathbf{Z}_n \cdot \cos(\omega \cdot h) + \mathbf{W}_n \cdot \sin(\omega \cdot h)) + F(x_n, y_n) \\ \exp(\alpha \cdot h) \cdot \mathbf{W}_n \cdot \cos(\omega \cdot h) - \mathbf{Z}_n \cdot \sin(\omega \cdot h)) + F(x_n, y_n) \end{pmatrix} \tag{17.55}$$

Rewriting the above equations in the terminology of [1] we have, with the following abbreviations

$$\mathbf{X}_n = ((x_n - Q(v)); \ \mathbf{Y}_n = ((y_n - Q(v)) \tag{17.56}$$

$$\mathbf{Z}_n = ((z_n - Q(u)); \ \mathbf{W}_n = ((w_n - Q(u)) \tag{17.57}$$

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \\ w_{n+1} \end{pmatrix} = \begin{pmatrix} \exp(\alpha \cdot h) \cdot \mathbf{X}_n \cdot \cos(\omega \cdot h) + \mathbf{Y}_n \cdot \sin(\omega \cdot h)) + Q(v) \\ \exp(\alpha \cdot h) \cdot \mathbf{Y}_n \cdot \cos(\omega \cdot h) - \mathbf{X}_n \cdot \sin(\omega \cdot h)) + Q(v) \\ \exp(\alpha \cdot h) \cdot \mathbf{Z}_n \cdot \cos(\omega \cdot h) + \mathbf{W}_n \cdot \sin(\omega \cdot h)) + Q(u) \\ \exp(\alpha \cdot h) \cdot \mathbf{W}_n \cdot \cos(\omega \cdot h) - \mathbf{Z}_n \cdot \sin(\omega \cdot h)) + Q(u) \end{pmatrix} \tag{17.58}$$

where

$$Q(s) = q_m \cdot \left(1 - \exp\left(\frac{1 - \exp(s)}{q_m} \right) \right) \tag{17.59}$$

and $v = x - 5.23 \cdot w \cdot z$ and $u = y - 0.1 \cdot x$ and $q_m = 5.0$

Again, we see that two second order ODEs are replaced by 4 IDs.

17.6 The KIII-ID Model Can Provide a Reduction in Computation as Well as Insights into the Neurodynamics

We now define the KIII-ID as follows. Assume that Eq. 17.60 is true.

$$\sum_{j}^{N} \sum_{\tau}^{T} k_{ijk} Q(y_j(t-\tau)) \approx Q(f(y_1, y_2, \dots y_N)) = \Phi(\mathbf{X}) \tag{17.60}$$

and let $\mathcal{Q}$ be defined as follows:

$$\mathcal{Q} = \sum_{j}^{N} w_j Q(y_j) \tag{17.61}$$

Let $\Psi(\mathbf{X}) = \mathcal{Q}(\mathbf{X}) + Q(f(y_1, y_2, \dots y_N)) = \mathcal{Q}(\mathbf{X}) + \Phi(\mathbf{X})$ Then KIII-ID is given by

$$\mathbf{T}_h(\mathbf{X}) = \exp(\mathbf{A} \cdot h)(\mathbf{X} - \Psi(\mathbf{X})) + \Psi(\mathbf{X}) \tag{17.62}$$

The mesoscopic theory requires a wave to pulse dynamic to communicate intent to local regions of the brain responsible for initiating action quickly. This wave dynamic may be what is referred to as a calcium wave in [12] that moves through the neuropil. This leads us to conjecture that the KIII-ID model can be further abstracted by the introduction of a wave/field concept. To arrive at the KIII-ID field model we make the following assumptions:

1. The KIII model was derived from the Neurodynamics of the brain using the simplest possible approach that captures the essential features of EEG studies.
2. The actual dynamics of the brain are so complex that it is reasonable to try to abstract from the KIII model only the essential concepts and dynamics inherent in that model assuming a field theory of the brain.
3. Then derive an abstract model of KIII by reverse engineering the KIII ODE model. To do this, two modification to the KIII theory were introduced: (A) in place of ODEs we used IDs that provide a dramatic simplification of the Runge-Kutta integration approach while retaining all dynamics and providing for step size variation without any loss of morphological accuracy. (B) Assume that the forcing function of KIII model, which was derived by direct experimentation, must be morphologically equivalent to a field force having a much simpler form.
4. $\Psi(f(\mathbf{X}))$ is the Field-composite of all interactions between nodes of the KII model. In terms of ID theory, f will represent the transition surface in n-dimensional space which governs the stretching wave in the neuropil. $\exp(\mathbf{A} \cdot h)$ will provide the folding wave component.

Given these abstractions, we present a simulation of the KII-ID model and the KI-ID model and contrast their morphology with EEG recordings from [14]. The

Table 17.1 Data for the KI-ID system in Fig. 17.1a

Damping and frequency	$\alpha = -0.1 : \beta = 0.5$
Step size	$h = 0.001$
Number of iterations	$N = 1,000,000$
Initial conditions	$x = 0 : y = 1 : z = 0 : w = 1.5$

Table 17.2 Parameters of the KII-ID model in Fig. 17.1b

Damping and frequency	$\alpha = -0.08 : \beta = 0.9$
Step size	$h = 0.001$
Number of iterations	$N = 1,000,000$
Initial conditions	$x = 0 : y = 1 : z = 0 : w = 1.5$
Initial conditions	$x_1 = 0 : y_1 = 1 : z_1 = 1 : w_1 = 1.5$

Table 17.3 Parameters for KIII-ID shown in Fig. 17.2

Damping and frequency	$\alpha = -0.03 : \beta = 0.5$
Step size	$h = 0.01$
Number of iterations	$N = 1,000,000$
Initial conditions	$x = 0 : y = 0.02 : z = 0 : w = 0.05$
Initial conditions	$x_1 = 0 : y_1 = 0.2 : z_1 = 0 : w_1 = 0.5$
Initial conditions	$x_2 = 0 : y_2 = 0.2 : z_2 = 0 : w_2 = 0.5$

simulations are an abstraction of Eqs. (3) and (13) from [1] (Tables 17.1, 17.2 and 17.3).

Iteration equations for KI-ID are as follows:

$$Q_0 = Q(x - 5.23 \cdot w \cdot z) \tag{17.63}$$

$$Q_1 = Q(y - 0.1 \cdot x) \tag{17.64}$$

$$Q(v) = 5.0 \cdot (1 - \exp((1 - \exp(v))/5.0))) \tag{17.65}$$

$$x \rightarrow \exp(\alpha \cdot h) \cdot ((x - Q_1) \cdot \cos(\beta \cdot h) + (y - Q_1) \cdot \sin(\beta \cdot h)) + Q_1 \tag{17.66}$$

$$y \rightarrow \exp(\alpha \cdot h) \cdot ((y - Q_1) \cdot \cos(\beta \cdot h) - (x - Q_1) \cdot \sin(\beta \cdot h)) + Q_1 \tag{17.67}$$

$$z \rightarrow \exp(\alpha \cdot h) \cdot ((z - Q_0) \cdot \cos(\beta \cdot h) + (w - Q_0) \cdot \sin(\beta \cdot h)) + Q_0 \tag{17.68}$$

$$w \rightarrow \exp(\alpha \cdot h) \cdot ((w - Q_0) \cdot \cos(\beta \cdot h) - (z - Q_0) \cdot \sin(\beta \cdot h)) + Q_0 \tag{17.69}$$
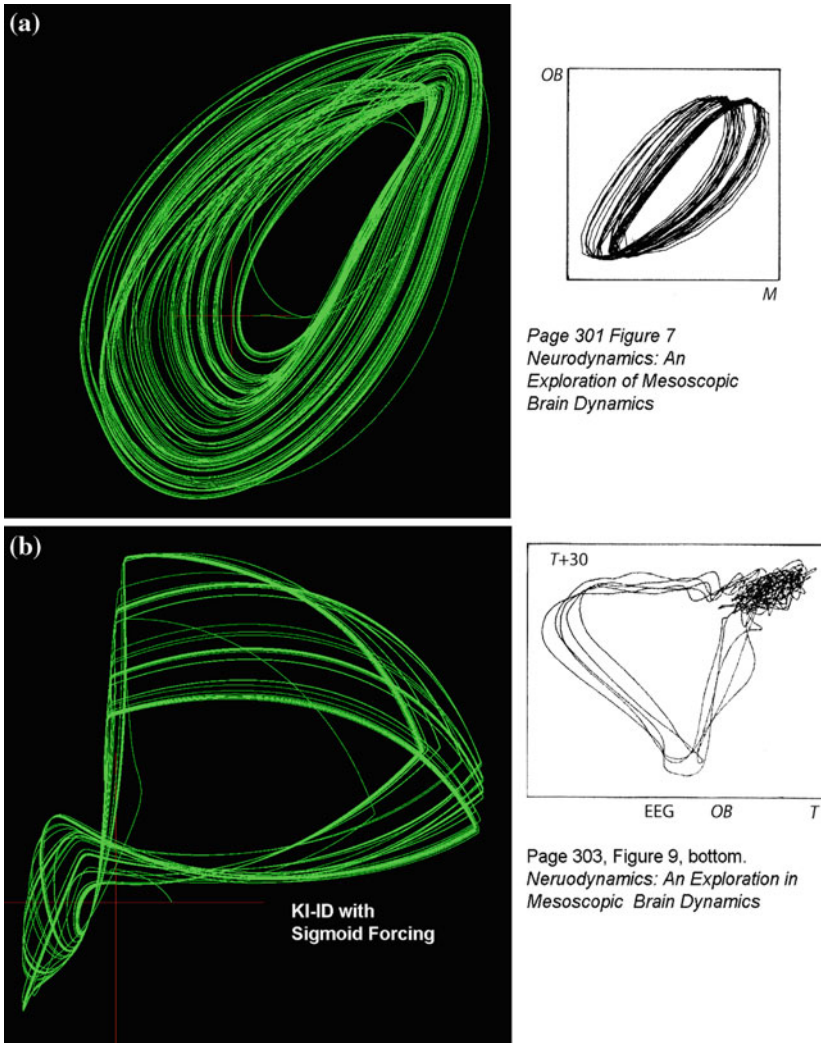
Fig. 17.1 Plate A Page 301, Fig. 7, *Left Plate* from Freeman [14], Plate B (p. 303), Fig. 9, *bottom*

409   Iteration equations for KII-ID are as follows:

$$Q_0 = Q(y_1) + 0.6 \cdot Q(z) \tag{17.70}$$

$$Q_1 = Q(y_1 + w_1) - Q(z) \tag{17.71}$$

$$Q_3 = Q(y - w) + 1.1 \cdot Q(z) \tag{17.72}$$

$$Q_4 = Q(y - x) \tag{17.73}$$

$$Q(v) = 5.0 \cdot (1 - \exp((1 - \exp(v))/5.0))) \tag{17.74}$$

$$x \rightarrow \exp(\alpha \cdot h) \cdot ((x - Q_1) \cdot \cos(\beta \cdot h) + (y - Q_1) \cdot \sin(\beta \cdot h)) + Q_1 \tag{17.75}$$

$$y \rightarrow \exp(\alpha \cdot h) \cdot ((y - Q_1) \cdot \cos(\beta \cdot h) - (x - Q_1) \cdot \sin(\beta \cdot h)) + Q_1 \tag{17.76}$$

$$z \to \exp(\alpha \cdot h) \cdot ((z - Q_0) \cdot \cos(\beta \cdot h) + (w - Q_0) \cdot \sin(\beta \cdot h)) + Q_0 \quad (17.77)$$

$$w \to \exp(\alpha \cdot h) \cdot ((w - Q_0) \cdot \cos(\beta \cdot h) - (z - Q_0) \cdot \sin(\beta \cdot h)) + Q_0 \quad (17.78)$$

$$x_1 \to \exp(\alpha \cdot h) \cdot ((x_1 - Q_3) \cdot \cos(\beta \cdot h) + (y_1 - Q_3) \cdot \sin(\beta \cdot h)) + Q_3 \quad (17.79)$$

$$y_1 \to \exp(\alpha \cdot h) \cdot ((y_1 - Q_3) \cdot \cos(\beta \cdot h) - (x_1 - Q_3) \cdot \sin(\beta \cdot h)) + Q_3 \quad (17.80)$$

$$z_1 \to \exp(\alpha \cdot h) \cdot ((z_1 - Q_4) \cdot \cos(\beta \cdot h) + (w_1 - Q_4) \cdot \sin(\beta \cdot h)) + Q_4 \quad (17.81)$$

$$w_1 \to \exp(\alpha \cdot h) \cdot ((w_1 - Q_4) \cdot \cos(\beta \cdot h) - (z_1 - Q_4) \cdot \sin(\beta \cdot h)) + Q_4 \quad (17.82)$$

17.7 The Wave $\Psi(X)$ for Any K Model May Arise from Partial Differential Equations that Must Be Derived from Experiment

While the sigmoid function is known to describe neuron binary dynamics, the complex summation of sigmoid functions could be replaced by a morphologically equivalent function which is known to satisfy a wave equation, for example $\sin(u) + \sin(3 \cdot u)/3 + \cdots$. Figure 17.2 compares using a wave-sigmoid dynamic to a Global wave dynamic in the KII-ID model.

In the KIII-ID model, the function $Q(v)$ which represents the transfer from a wave to an impulse is replaced by a new function that collectively describes the local dynamics without considering the specific dynamics of wave-pulse interaction. This is a mathematical abstraction and simplification to is a break from the physics and a transition to just to the collective dynamics of all forces and interactions combined. Making this abstraction alleviates the researcher unskilled in neuroscience from fully understanding the particulars of the wave to pulse dynamic and only considering to mathematical dynamics. While this does place the engineer a step away from the neuroscience, it may also facilitate formulations that will encompass additional insights and provide access to the KIII theory by scientists and engineers unskilled in the details of neuroscience.

Iteration equations for KIII-ID are as follows:

$$Q_0 = Q(w) + Q(x_2 + \Psi(x_2, w_1, z, w) \quad (17.83)$$

$$Q_1 = Q(y_1 + w_1) - Q(z) \quad (17.84)$$

$$Q_3 = Q(y - w) + 1.1 \cdot Q(z) \quad (17.85)$$

$$Q_4 = Q(y - x) \quad (17.86)$$

$$Q(v) = (1 - \exp((1 - \exp(v)))) \quad \text{Plate A} \quad (17.87)$$

$$Q(v) = \sin(v) + \sin(3 \cdot v)/3 \quad \text{Plate B} \quad (17.88)$$

$$\Psi(x, y, z, w) = \exp(\alpha \cdot x) \cdot \cos(\alpha \cdot y)$$

$$+ \sin(5 \cdot z) \cdot \cos(\cos(x) \cdot w) \quad (17.89)$$

$$x \to \exp(\alpha \cdot h) \cdot ((x - Q_1) \cdot \cos(\beta \cdot h) \quad (17.90)$$

$$+ (y - Q_1) \cdot \sin(\beta \cdot h)) + Q_1 \quad (17.91)$$

$$y \to \exp(\alpha \cdot h) \cdot ((y - Q_1) \cdot \cos(\beta \cdot h) \quad (17.92)$$

Plate A

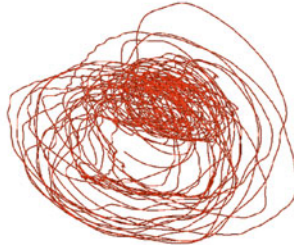ex1 = (1 - Exp(v)) / qm
Q(v) = qm (1 - Exp(ex1))

**KIII-ID
Sigmoid Model**

Plate B

Q(v)=sin(v)+sin(3v)/3

**KIII-ID
Global Wave
Model**

Fig. 17.2 KIII-ID Sigmoid model versus the KIII-ID wave model

$$- (x - Q_1) \cdot \sin(\beta \cdot h)) + Q_1 \tag{17.93}$$

$$z \to \exp(\alpha \cdot h) \cdot ((z - Q_0) \cdot \cos(\beta \cdot h) \tag{17.94}$$

$$+ (w - Q_0) \cdot \sin(\beta \cdot h)) + Q_0 \tag{17.95}$$

$$w \to \exp(\alpha \cdot h) \cdot ((w - Q_0) \cdot \cos(\beta \cdot h) \tag{17.96}$$

$$- (z - Q_0) \cdot \sin(\beta \cdot h)) + Q_0 \tag{17.97}$$

$$x_1 \to \exp(\alpha \cdot h) \cdot ((x_1 - Q_3) \cdot \cos(\beta \cdot h) \tag{17.98}$$

$$+ (y_1 - Q_3) \cdot \sin(\beta \cdot h)) + Q_3 \tag{17.99}$$

$$y_1 \to \exp(\alpha \cdot h) \cdot ((y_1 - Q_3) \cdot \cos(\beta \cdot h) \tag{17.100}$$

$$- (x_1 - Q_3) \cdot \sin(\beta \cdot h)) + Q_3 \tag{17.101}$$

$$z_1 \to \exp(\alpha \cdot h) \cdot ((z_1 - Q_2) \cdot \cos(\beta \cdot h) \tag{17.102}$$

$$+ (w_1 - Q_2) \cdot \sin(\beta \cdot h)) + Q_2 \tag{17.103}$$

$$w_1 \to \exp(\alpha \cdot h) \cdot ((w_1 - Q_2) \cdot \cos(\beta \cdot h) \tag{17.104}$$

$$- (z_1 - Q_2) \cdot \sin(\beta \cdot h)) + Q_2 \tag{17.105}$$

$$x_2 \to \exp(\alpha \cdot h) \cdot ((x_2 - Q_5) \cdot \cos(\beta \cdot h) \tag{17.106}$$

$$+ (y_2 - Q_5) \cdot \sin(\beta \cdot h)) + Q_5 \tag{17.107}$$

$$y_2 \rightarrow \exp(\alpha \cdot h) \cdot ((y_2 - Q_5) \cdot \cos(\beta \cdot h) \tag{17.108}$$

$$- (x_2 - Q_5) \cdot \sin(\beta \cdot h)) + Q_5 \tag{17.109}$$

$$z_2 \rightarrow \exp(\alpha \cdot h) \cdot ((z_2 - Q_4) \cdot \cos(\beta \cdot h) \tag{17.110}$$

$$+ (w_2 - Q_4) \cdot \sin(\beta \cdot h)) + Q_4 \tag{17.111}$$

$$w_2 \rightarrow \exp(\alpha \cdot h) \cdot ((w_2 - Q_4) \cdot \cos(\beta \cdot h) \tag{17.112}$$

$$- (z_2 - Q_4) \cdot \sin(\beta \cdot h)) + Q_4 \tag{17.113}$$

17.8 Summary

Starting with the KIII wave theory of Freeman-Kozma, we derived a top-down mathematical model, KIII-ID, which used the concept of stretching and folding in place of the laws of physics. We noted that the ID model has a mathematical foundation that has broad applicability to many dynamical systems including the KIII ODEs. We discussed some of the simplifying advantages of the KIII-ID approach and then we used the KII-ID model to morphologically replicate results from the KIII model of known EEGs. Finally we suggested that the sigmoid function could be replaced by solutions of wave equations which may lead to further simplifications of the KIII theory and make it more accessible to researchers without an extensive knowledge of neurodynamics as well as more amenable to formal scientific proof.

References

1. Ilin R, Kozma R (2006) Stability of coupled excitatory-inhibitory neural populations and applications to control of multi-stable systems. Phys Lett A 360:66–83
2. Smale S (1967) Differentiable dynamical systems. Bull Am Math Soc 73:747–817
3. Hénon M (1976) A two-dimensional mapping with a strange attractor. Commun Math Phys 50(1):69–77
4. Brown R (2014) The Hirsch conjecture. Dynamics of continuous, discrete and impulsive systems (to appear)
5. Brown R (2015) The theory of infinitesimal diffeomorphisms: an introduction (to appear)
6. Brown R, Jain V (2009) A new approach to Chaos. Dyn Contin Discret Impuls Syst Ser A Math Anal 16:863–890
7. Hirsch MW (1985) Fischer P, Smith W (eds) The chaos of dynamical systems, chaos, fractals and dynamics. Marcel Decker, Inc, New York
8. Brown R (2015) Stretching and Folding Transitions in the Infinitesimal Hénon Diffeomorphism (to appear)
9. Brown R, Chua L (1996) From almost periodic to chaotic: the fundamental map. Int J Bifurc Chaos 6(6)
10. Brown R (2015) A short history of stretching and folding (to appear)
11. Brown R (2014) Infinitesimal stretching and folding II: the simple scroll. Dyn Contin Discret Impuls Syst (to appear)
12. Brown R (2014) Infinitesimal stretching and folding I (to appear)
13. Brown R (1992) Generalizations of the Chua equations. Int J Bifurc Chaos 4:889–909
14. Freeman WJ (2000) Neurodynamics: an exploration in mesoscopic brain dynamics. Springer, New York

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Abstract	In this work, we examine the basic problem of connecting a theory of the brain to the activities of humans engaged in the common pursuits of everyday life. This examination is explored through an implementation of a current dynamical theory, the KIII theory, which originates with Freeman (Neurodynamics: an exploration in mesoscopic brain dynamics, 2000, [4]) and is advanced mathematically by Kozma. Our venue is a sports training program which is chosen for its accessibility to all researchers. In order to carry out this examination we must use a mathematical framework that serves the purpose of capturing the dynamics of the Freeman-Kozma model (Freeman, Neurodynamics: an exploration in mesoscopic brain dynamics, 2000, [4], Ilin and Kozma, Phys Lett A 360:66–83, 2006, [7]) and which can also be applied to the activities of a human enterprise.