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CHAOS OR TURBULENCE?

RAY BROWN and LEON O. CHUA
Department of Electrical Engineering and Computer Sciences, University of California, Berkeley, California 94720, USA

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We present a new three-dimensional autonomous chaotic dynamical system that *appears* to have a closer relationship to turbulence than the Lorenz system. We have developed this system using the new technique of *dynamical synthesis.*

1. Introduction

There is considerable belief that chaos and turbulence are closely related. Because we share this position, we have constructed an example through the method of *dynamical synthesis* (explained in Sec. 3) to make the relationship between chaos and turbulence more convincing. From the construction it is clear how to extend our example to form a much larger dynamical system that could have many characteristics of fluid flow. Section 2 presents our construction of what we are calling a *vortex* equation. Section 3 elaborates on the general technique of dynamical synthesis which can be used to construct dynamical systems having given properties.

2. Synthesis of a Vortex

In this section we synthesize a system which is a simplified version of a turbulent vortex with convection. The trajectories of this dynamical system will swirl down the vortex until they reach the bottom where they will leave and then reenter the top of the vortex. Once at the top, the process will begin over again. The appearance of the attractor in this example is that of a simplified tornado.

The first thing we need is a component which is a two-dimensional dynamical system that rotates around a fixed point in such a manner that as we approach the fixed point, the rate of rotation increases. We can obtain this from a twist map with rotation function $f(r) = 1/r$, [Brown & Chua, 1991]. The equation for this ODE is

$$\dot{x} = -\omega f(r)(y - b_1)\,,$$
$$\dot{y} = -\omega f(r)(x - a_1)\,,$$

where (a_1, b_1) is the fixed point, and

$$r = \sqrt{(x - a_1)^2 + (y - b_1)^2}\,.$$

Next, we add damping so that the fixed point (a_1, b_1) is attracting. Doing this we have the autonomous equation

$$\dot{x} = -\alpha(x - a_1) - \omega f(r)(y - b_1)\,,$$
$$\dot{y} = -\alpha(y - b_1) + \omega f(r)(x - a_1)\,.$$

We now combine this equation with a one-dimensional autonomous system to obtain the following three-dimensional system which provides a simplified vortex centered at (a_1, b_1, c_1):

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{bmatrix} -\alpha & -\omega/r & 0.0 \\ \omega/r & -\alpha & 0.0 \\ 0.0 & 0.0 & \gamma \end{bmatrix} \begin{pmatrix} x - a_1 \\ y - b_1 \\ z - c_1 \end{pmatrix} \qquad (1)$$

In this equation, initial conditions at the top of the vortex produce trajectories that spin down the vortex to infinity. Figure 1 illustrates an orbit of Eq. (1) in the x–z plane.

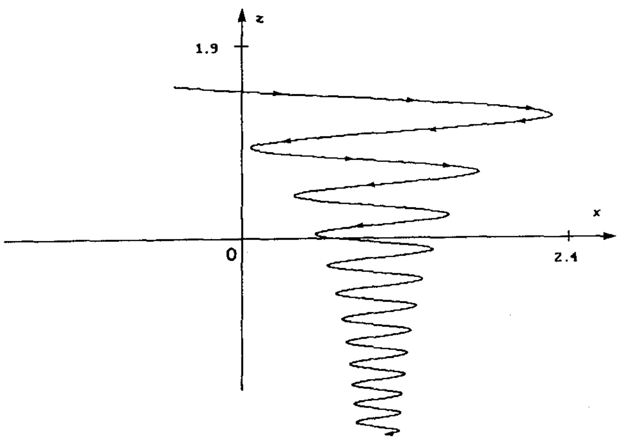

Fig. 1. Orbit of Eq. (1) in the x–z plane.

We need some means of stopping these trajectories after a finite distance and returning them to the top of the vortex. Thus what we need is a lower boundary and then a device at this boundary that is similar to convection.

We will use a very simplistic equation for the convection component in this example in order to keep the illustration easy to follow, but it should be noted that a more accurate example can be used and the process would not change. Our *simplistic* convection is given by the following linear autonomous ODE:

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{bmatrix} -\beta_0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & \beta_0 \end{bmatrix} \begin{pmatrix} x - a_2 \\ y - b_2 \\ z - c_2 \end{pmatrix} . \quad (2)$$

Figure 2 illustrates the effect of the dynamical system defined by Eq. (2) in the x–z plane.

We now join the vortex equation with the simplified convection equation across a transition plane in a manner similar to that done to synthesize the Lorenz equations [Brown, 1992]. There are many choices for a plane that can be used to separate these two dynamical systems but for simplicity the plane we use is given by $z + 2x = 0$. Putting all of this together, the final equation is given by

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = s(u)A_1 \begin{pmatrix} x - a_1 \\ y - b_1 \\ z - c_1 \end{pmatrix} + (1 - s(u))A_2 \begin{pmatrix} x - a_2 \\ y - b_2 \\ z - c_2 \end{pmatrix} , \quad (3)$$

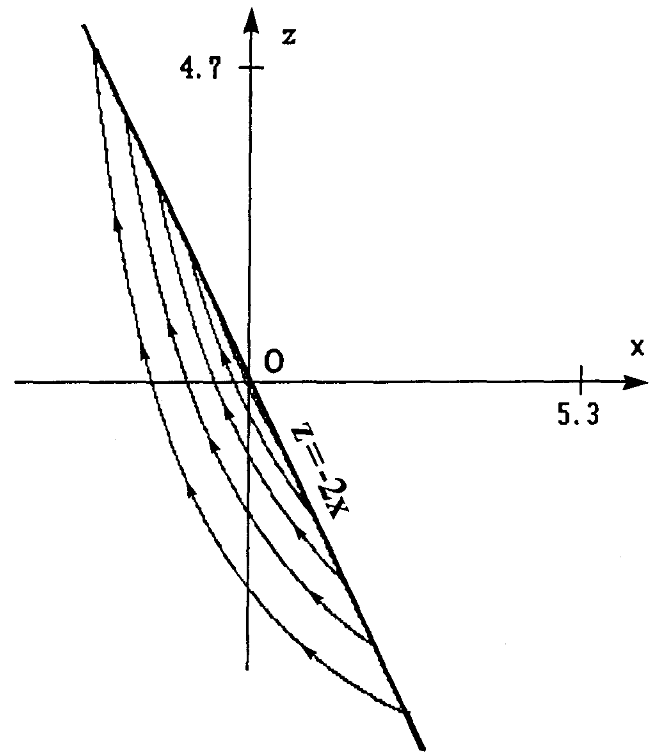

Fig. 2. Simplified convection: orbits of Eq. (2) in the x–z plane to the left of the line $z = -2x$.

where, $s(u) = 0.5[1 + \mathrm{sgn}(u)]$. The function $s(u)$ is generally known as the unit step function or the Heaviside function.

$$A_1 = \begin{bmatrix} -\alpha & -\omega/r & 0.0 \\ \omega/r & -\alpha & 0.0 \\ 0.0 & 0.0 & \gamma \end{bmatrix} ,$$

and

$$A_2 = \begin{bmatrix} -\beta_0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & \beta_0 \end{bmatrix} ,$$

$u = 2x + z$, and $r = \sqrt{(x - a_1)^2 + (y - b_1)^2}$.

Figure 3(a) illustrates a short trajectory of Eq. (3) in the x–z plane.

Figure 3(b) shows a short trajectory for Eq. (3) in three-dimensions. Figure 4 shows a longer trajectory for Eq. (3), indicating the attractor. As noted in Brown [1992] the function sgn(u) used in the definition of $s(u)$ can be replaced by $\tanh(\beta u/2)$ and the equation is made C^∞.

We concede that this example is simple; however it can be made systematically more accurate by partitioning the three-dimensional space, $\mathbf{R}^3$, into more regions and extending the model further to

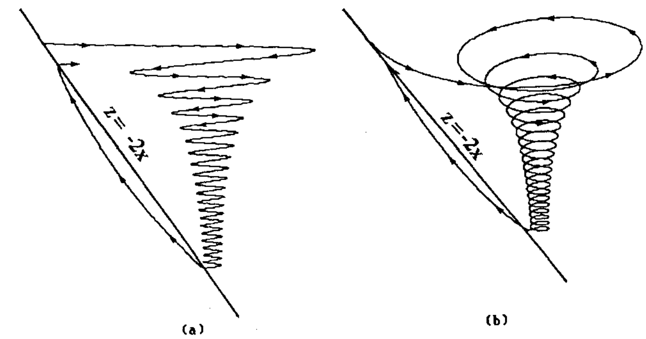

Fig. 3. Short trajectory for the synthesized vortex: (a) in the x–z plane; (b) in the x–y–z space.

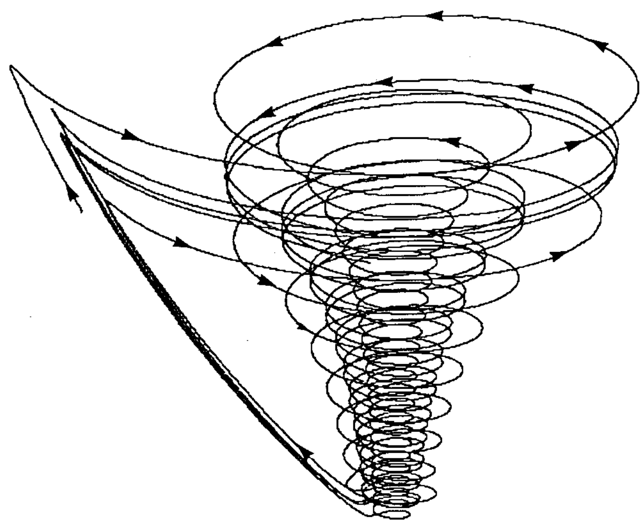

Fig. 4. Attractor for the synthesized vortex.

include more features of the atmosphere. Also, by making the point (a_1, b_1, c_1) a slowly-varying function of time, our static vortex will begin to move accordingly. And as we stated earlier, our simplistic convection model can be replaced by a more accurate model of convection without altering the principles of our *synthesis* procedure.

3. Dynamical Synthesis

The technique of dynamical synthesis[1] has evolved over the past three years. Its development has been guided by our long standing belief that there is a pressing need for new nonlinear techniques that:

- Are constructive in nature and do not require a nonlinear scientist to have sophisticated mathematical knowledge or insight to apply.
- Employ a building block approach whereby simple well-understood components are used to construct models of complex dynamical systems.
- Utilize two-dimensional autonomous systems and n-dimensional linear systems as building blocks.
- Reduce the reliance on classical numerical analysis and therefore reduce reliance on super computers now needed to numerically integrate complex dynamical models, such as the Navier–Stokes equations and its variants.

3.1. *Discussion of dynamical synthesis*

The methods of global analysis originated by Poincaré and carried on by Birkhoff, Smale, Arnold, Hirsch, Pugh, and others have pointed us toward the geometric, or global, view of dynamical systems wherein algebraic formulas and deviations are replaced by geometric concepts and abstract arguments as the tools of analysis. However, geometry and abstraction in the absence of algebraic formulas and derivations are not enough to provide the tools needed by nonlinear scientists. Abstractions and geometry must still be resolved or reduced to explicit formulas from which we can construct models and theories, [Boyd & Chua, 1985], [Chua 1971], [Chua & Schilling, 1974], [Chua & Green, 1974], [Chua & Kang, 1978], [Chua & Stromsmoe, 1971].

Consistent with constructive methods, a theory of nonlinear dynamical systems must be able to tell us how to decompose complex dynamical processes into simple parts or building blocks. Ideally, any theory of nonlinear dynamics should show how to reduce the construction of complex dynamical systems to building blocks composed of solutions of second-order autonomous ordinary differential equations and/or to general linear ordinary differential equations.

Following the global analyst, we organize our thinking about nonlinear problems by an analysis of its periodic points and preferably, its fixed points. But we add another factor based on experience with, and observation and simulation of, nonlinear systems: Each fixed point has a "region of significance" outside of which some other fixed point takes over. If we can determine both the fixed points and their regions of significance we have the beginnings of a decomposition of the global dynamical system into more manageable parts. It is essential from this point of view that after the determination of the fixed points and their regions of significance, we also determine a decomposition of the dynamical

[1]The term "synthesis" is motivated by a similar use of the term in other fields.

system at each fixed point into components, which are individually well understood, such as linear autonomous, or two-dimensional autonomous, nonlinear systems, or into Morse–Smale systems. Ideally this decomposition is into one and two-dimensional systems. The boundaries of the regions of significance are hopefully simple planes or other algebraic surfaces for which there are formulas. Our ability to carry out this determination is predicated on the fact that, in a great many of the complex dynamical systems we encounter in nonlinear science today, *the source of complexity in the dynamical system arises from the manner in which various local, simple, and well-understood, nonlinear components fit together to form the larger global dynamical system.* This may be called the principle of dynamical synthesis. Once these regions are determined, we can decompose and synthesize the dynamical system in question into its significant parts and their regions of significance, and it only remains to determine how the interplay of the separate well-known parts across the boundaries of these regions combines to create the global dynamics.

Our goal is to characterize the global dynamics of the original system by the construction of explicit and simple formulas whose terms are formed from the definitions of the regions and the well-known dynamical systems within these regions. This process leads to a "factorization" of the dynamical system related to Brown & Chua [1991] into known and well-understood components from which Poincaré maps may be obtained in closed form, or which may be reduced to one-dimensional maps.

It is interesting that this mathematical approach has been intuitively suggested by various scientists working in the area of neural science as well as other disciplines. The olfactory model of Freeman [Yao & Freeman, 1990] and the cellular neural network of Chua [Chua & Yang, 1985] are particularly relevant examples.

The above considerations combined with extensive experience leads to the following statement of the two fundamental questions of dynamical synthesis:

(1) When can we synthesize the most important global properties of a nonlinear dynamical system from the dynamics of the system that occurs within a region of a subset of its periodic points, or better, its fixed points?
(2) When can the local dynamics of a system be constructed from well-known and well-understood one- and two-dimensional dynamical systems or from n-dimensional linear autonomous systems?

The answers to these two questions parallel in an interesting way two historically important questions of harmonic analysis: that of spectral analysis and spectral synthesis of signals [Rudin, 1962; Katznelson, 1968; Graham & McGehee, 1979]. In place of seeking dominant frequencies, i.e. spectra, we seek dominant periodic points and their regions of significance; this is the analysis phase. Next we seek to reconstruct (synthesize) the most important global features of a nonlinear dynamical system (signal) from its dominant periodic points (frequencies) and their regions of significance. As in harmonic analysis, we want to carry out this analysis and synthesis in terms of the simplest possible dynamical components. In harmonic analysis these components are the sines and cosines or other wavelets; in dynamics these components are the most well-understood one and two-dimensional autonomous or n-dimensional linear autonomous dynamical systems.

The significance of dynamical synthesis is that it puts into the hands of the working scientist simple tools from which she or he can synthesize or model her or his particular dynamical system from simple, well-understood building blocks.

The objective of dynamical synthesis is to derive a model of a dynamical system which is (1) simple enough to admit some practical form of analysis, (2) sophisticated enough to contain the most important global features of interest to the scientist, and (3) to have a bifurcation or perturbation structure similar to the original system. Such synthesis may allow us to determine how the global dynamics of a system changes as the parameters of the system change and, while we are unable to predict the future of any particular time series, we hope to predict the dynamical impact that the parameters have on a system. An interesting example of a problem that dynamical synthesis might be useful in studying is to analytically determine how local geophysical changes in our environment result in long-term global changes in our atmosphere.

Of interest in this regard is the work being carried out in the area of embedding where it is desired to discover global dynamics of a system by considering only the measurements of a single observable time series [Takens, 1981]. The method of dynamical synthesis may be applied to the reconstructed *attractor*, or embedded time series to obtain a global analytical model of the dynamical system.

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References

Boyd, S. & Chua, L. [1985] "Fading memory and the problem of nonlinear operators with Volterra series," *IEEE Trans. Circuits and Systems* **32**, 1150–1161.

Brown, R. [1992] "Generalizations of the Chua Equations," *Int. J. Bifurcation and Chaos* **2**(4), 889–909.

Brown, R. & Chua, L. [1991] "Generalizing the Twist-and-Flip Paradigm," *Int. J. Bifurcation and Chaos* **1**(2), 385–416.

Chua, L. O. [1971] "Synthesis of nonlinear systems with prescribed singularities," *IEEE Trans. Circuit Theory* **18**, 375–382.

Chua, L. O. & Schilling, R. J. [1974] "An algorithm for modeling the sinusoidal input/steady state response behavior of nonlinear systems over a set of frequencies and amplitudes," *J. Franklin Institute* **298**, 101–124.

Chua, L. O. & Green, D. N. [1974] "Synthesis of nonlinear periodic systems," *IEEE Trans. Circuits and Systems* **21**, 286–294.

Chua, L. O. & Kang, S. M. [1978] "Memristive devices and systems," *Proc. of the IEEE* **4**, 209–223.

Chua, L. O. & Stromsmoe, K. A. [1971] "Mathematical models for dynamic hysteresis loops," *Int. J. Engineering Science* **9**, 435–450.

Chua, L. O. & Yang, L. [1988] "Cellular Neural Networks: Theory and Applications," *IEEE Trans. Circuits and Systems* **35**, 1257–1290.

Graham, C. C. & McGehee, O. C. [1979] *Essays in Commutative Harmonic Analysis* (Springer-Verlag, New York).

Katznelson, Y. [1968] *An Introduction to Harmonic Analysis* (Dover, New York).

Rudin, W. [1962] *Fourier Analysis on Groups* (Interscience Publishers, New York).

Takens, F. [1981] "Detecting strange attractors in turbulence," *Lecture Notes in Mathematics: Dynamical Systems and Turbulence, Warwick 1980* (Springer-Verlag, Berlin).

Yao, Y. & Freeman, W. J. [1990] "Model of Biological Pattern Recognition with Spatially Chaotic Dynamics," *Neural Networks* **3**, 153–170.