International Journal of Bifurcation and Chaos, Vol. 11, No. 1 (2001) 19–26

CHAOS AND COMPLEXITY

RAY BROWN and ROBERT BEREZDIVIN
Raytheon Systems Company, Falls Church, VA, USA

LEON O. CHUA
Department of Electrical Engineering and Computer Sciences,
University of California, Berkeley, 94720, USA

Received February 20, 2000; Revised June 15, 2000

In this paper we show how to relate a form of high-dimensional complexity to chaotic and other types of dynamical systems. The derivation shows how "near-chaotic" complexity can arise without the presence of homoclinic tangles or positive Lyapunov exponents. The relationship we derive follows from the observation that the elements of invariant finite integer lattices of high-dimensional dynamical systems can, themselves, be viewed as single integers rather than coordinates of a point in n-space. From this observation it is possible to construct high-dimensional dynamical systems which have properties of shifts but for which there is no conventional topological conjugacy to a shift. The particular manner in which the shift appears in high-dimensional dynamical systems suggests that some forms of complexity arise from the presence of chaotic dynamics which are obscured by the large dimensionality of the system domain.

1. Introduction

The study of chaos has raised many interesting questions about highly complicated nonlinear systems generally. This is because, in an effort to answer questions about chaos, it has been necessary to undertake the study of nonchaotic processes such as skew translations and infinite dimensional rotations which can produce dynamics that can appear related to chaos. The nonchaotic strange attractor is one example. The study of dynamics at the *edge* of chaos that has resulted from investigations into complex nonchaotic systems has led to the study of complexity as a separate discipline. Thus we arrive at the question: What is the relationship between chaos and complexity? In this paper we provide a discussion of one possible bridge between these two important areas of research.

1.1. *Background*

Phillip Anderson, in [Anderson, 1994], describes eight disciplines that either have been, or are presently, making a contribution to the study of complexity. They are, roughly: (1) the theory of complexity or computability ala Chaiten, Kolomogrov, Church and others; (2) information theory; (3) ergodic theory and dynamical systems; (4) cellular automata and artificial life; (5) large random physical systems; (6) self-organized criticality; (7) artificial intelligence; and (8) neuroscience. Gell-Mann, Crutchfield, and their associates contribute to (1), (3)–(7), and there is also a long list of other contributors. In general, transitioning from these individual disciplines to rigorous mathematical theorems about complexity appears to be difficult. It is on this point that we present this paper. Specifically, we present a rigorous connection between complexity, chaos and various forms of complicated dynamics that have been studied extensively. Our connection proceeds through area (3) cited above.

1.2. *Notation*

Let T be a transformation on $\mathbf{R}^n$ that preserves a special finite subset which we will call $\mathcal{V}^n$. We define this subset as follows:

$$\mathcal{V}^n = \{\mathbf{x} \in \mathbf{R}^n | \mathbf{x} = (a_1, a_2, \ldots, a_n),\ a_i \in \{0, 1\}\}$$

As we have required that T preserves this set, we have $\mathrm{T}(\mathcal{V}^n) \subseteq \mathcal{V}^n$. We allow that T, under iteration, may map $\mathcal{V}^n$ to a proper subset of itself.

It is our objective to show that a certain class of transformations which preserve $\mathcal{V}^n$ can produce very complex dynamics. As an aid to seeing how complex dynamics can occur in this setting, we construct an invertible mapping of $\mathcal{V}^n$ into the unit interval I. This is done as follows: Let $X = (x_1, x_2, x_3, \ldots, x_n) \in \mathcal{V}^n \subset \mathbf{R}^n$. We define the invertible mapping $\pi : \mathcal{V}^n \to \mathrm{I}$ as

$$\pi(X) = (0.x_1x_2x_3, \ldots, x_n)$$

For example, let $(1, 1, 0, 0, 1) \in \mathcal{V}^n$, then $\pi(1, 1, 0, 0, 1) = 0.11001$.

For any high-dimensional dynamical system T, having a finite invariant subset of the type $\mathcal{V}^n$, the mapping π can be used to define a mapping on I that relates T to a one-dimensional map of I. In particular, if we define $S_{\mathrm{T}}(X) \equiv \pi(\mathrm{T}(\pi^{-1}(X)))$, then

$$S_{\mathrm{T}}(0.x_1x_2x_3, \ldots, x_n) = \pi(\mathrm{T}(x_1, x_2, x_3, \ldots, x_n))$$

Thus, when T is restricted to $\mathcal{V}^n$, T is conjugate to a one-dimensional map of a finite subset of I to itself. S_{T} may be conveniently viewed as a mapping on a subset of fractions between 0 and $(2^n - 1)/2^n$ in order to facilitate graphical representation of the one-dimensional map. This is expressed by transforming the point $0.x_1x_2x_3, \ldots, x_n$ by the formula

$$(0.x_1x_2x_3, \ldots, x_n) \to \sum_{i=1}^{n} \frac{x_i}{2^i}$$

We will use S_{T} to mean either the decimal or binary mapping so long as it is not ambiguous.

The mapping S_{T}, as we have defined it, is determined only on a finite subset of I by the mapping T. It can be extended to be a continuous mapping in any number of ways, all of which are equivalent so long as we are only examining the dynamics of T on $\mathcal{V}^n$.

Example. Define T as

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x + y - x \cdot y \\ 1 - x \end{pmatrix}$$

Then S_{T} is defined for four points in I. They are 0.0, 0.25, 0.5, 0.75. $S_{\mathrm{T}}(0.0) = 0.25$, $S_{\mathrm{T}}(0.25) = 0.75$, $S_{\mathrm{T}}(0.5) = 0.5$, $S_{\mathrm{T}}(0.75) = 0.5$. Since S_{T} is not defined for the point $(1, 0)$ by T, we extend S_{T} to include the interval $(0.75, 1.0)$ by assigning it to be the identity on this subinterval. As the dimension of T increases, this subinterval, which for an n-dimensional map is $((2^n - 1)/2^n, 1.0)$, goes to zero and so this convention is both harmless and useful.

It is clear that there are $n!$ representations of a given T, since the coordinates of T have $n!$ permutations. Thus there are $n!$ different ways of graphing T on I. However, the dynamics of all representations are qualitatively the same. For example, if a point converges to a given attractor in one representation, then it converges to an attractor of the same size in all representations. Thus, the dynamics of T are invariant with regard to the representation. It may happen, however, that one graphical representation may be more appropriate than another for a given T. We will see an example of this in the next section. For convenience, we will refer to π as the standard representation based on a given labeling of the coordinates of T.

In the next section we will show how to define n-dimensional maps which preserve $\mathcal{V}^n$ and have nearly chaotic dynamics. Before that, we give some examples of how to construct mappings that just preserve $\mathcal{V}^n$.

Example. Consider the following transformation on $\mathbf{R}^4$:

$$\begin{aligned} &\mathrm{T}(x, y, z, w) \\ &\quad = (x \cdot y \cdot z,\ x + y - x \cdot y,\ 1 - z \cdot x,\ x + y - 2x \cdot y) \end{aligned}$$

which preserves $\mathcal{V}^4$. We are able to construct such transformations generally by the requirement that each coordinate of the transformation defines a mapping of $\mathcal{V}^n$ to $\{0, 1\}$. The following are some examples of such functions:

$$\begin{aligned} (x, y, z) \to x + y + z - 2(x \cdot y + y \cdot z + z \cdot x) \\ + 3x \cdot y \cdot z \end{aligned}$$

$$(x, y, z) \to x \cdot y + y \cdot z + z \cdot x - 2x \cdot y \cdot z\,.$$

In the first example, if either of x, y, z is one, the result is one, otherwise it is zero. For example, if both of x, y are one, the result is zero. In the second example, at least two coordinates must be 1 for the result to be 1, otherwise it is zero.

We note that we may increase the exponent of any factor without losing the invariance of $\mathcal{V}^n$ since 1 or 0 to any power is still 1 or 0, respectively. Thus

$$(x,\, y,\, z) \to x^3 \cdot y + y \cdot z + z \cdot x - 2x \cdot y^5 \cdot z^{0.5}$$

also defines a mapping of $\mathcal{V}$ to the set $\{0,\, 1\}$.

2. Construction of Complex Dynamics in High-Dimensional Spaces

Using the notation of the previous section, we can generate complex orbits in high-dimensional spaces with very long periods by writing down a transformation that is conjugate to $3x \bmod (1)$ when restricted to $\mathcal{V}^n$. The map $3x \bmod (1)$ is chosen as it is able to generate complex orbits from points which have zeros in every coordinate position except 1. This would not be possible using $2x \bmod (1)$. To define T to be conjugate to $3x \bmod (1)$, T must carry out the function of a coordinate shift followed by a binary addition operation. The ith coordinate function for the shift-and-add transformation, T, is given by:

$$\begin{aligned} x_{i+1} &= y_i + v_{i-1} \cdot (1 - 2 \cdot (x_i + x_{i-1}) + 4 \cdot z_i) \\ y_i &= x_i + x_{i-1} - 2 \cdot z_i \\ z_i &= x_i \cdot x_{i-1} \\ v_i &= z_i + y_i \cdot v_{i-1} \end{aligned} \tag{1}$$

The equation for v_i does not have to be recursive in v_{i-1} and can be written out completely in advance. For the kth coordinate we have:

$$\begin{aligned} v_k = z_k &+ y_k \cdot z_{k-1} + y_k \cdot y_{k-1} \cdot z_{k-2} \\ &+ y_k \cdot y_{k-1} \cdot y_{k-2} \cdot z_{k-3} \\ &+ \cdots + y_k \cdot y_{k-1} \cdot y_{k-2} \cdots y_1 \cdot v_0 \end{aligned}$$

To start this equation, we must choose $v_0 = 0.0$ and so the last term drops out of the equation for v_i. As T is precisely a shift and add with carry on $\mathcal{V}^n$ it is conjugate to $3x \bmod (1)$ by direct inspection of the formulae. Figure 1 is a graph of S_{T} on the unit interval which will be recognized as the graph of $3x \bmod (1)$. This graph is constructed by assuming x_1 is in the first decimal place, i.e. we are using the standard representation, π. Any other representation would make it difficult to observe that T is actually $3 \cdot x \bmod (1)$ on $\mathcal{V}^n$. However, the fact that T is conjugate to a chaotic map on I is still clear from every representation of the coordinates of T as a number on the unit interval.

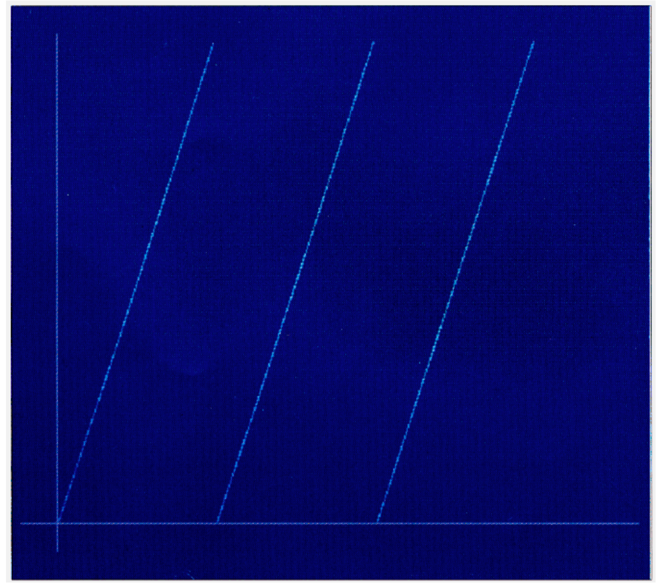

Fig. 1. In this figure we show the one-dimensional map, $S_{\mathrm{T}}(X)$ corresponding to Eq. (1). It is equal to the one-dimensional map $x \to 3x \bmod (1)$ on the domain of definition.

Some important observations about the shift-and-add transformation are: It can be defined in any number of dimensions. When T is restricted to points whose coordinates are all less than 1, it converges to the origin as a local attractor. For values greater than 1, it is unbounded. An interesting feature of T is that, as a dynamical system, it would not be readily recognized that T is conjugate to a unilateral shift on a finite subset. This is a key point. Very high-dimensional dynamical systems having finite invariant subsets can have a hidden complexity in the sense that the complexity is not readily apparent from the definition of the system. Because dynamical systems such as T may be equated to finite automata, T demonstrates that finite automata can have a degree of complexity whose source is obscured by the presence of its numerous dimensions or cells.

The conjugacy between T and S_{T} may be viewed in two ways. First, the orbits of T in $\mathcal{V}^n$ define orbits of S_{T} on I and so the attractors of T correspond to attractors of S_{T}. Second, since S_{T} is a function, it can be graphed. This is done by making a random selection of the points in $\mathcal{V}^n$ and plotting the value of S_{T}. This *Monte Carlo* method of graphing S_{T} is adequate for our purposes.

By composing T with various other transformations we may make the relationship between T and $3x \bmod (1)$ obscure. One interesting transformation is reflection through the center coordinate,

$x_i \to x_{n-i+1}$, where n is the dimension of the domain of T. The result of this composition is illustrated in Fig. 2.

Figure 2 is the graph of a function on the unit interval. The appearance of this graph could suggest the orbit of a point under the action of a two-dimensional map such as the cat map, [Arnold & Avez, 1968]. This illusion results from the high degree of oscillation of the graph.

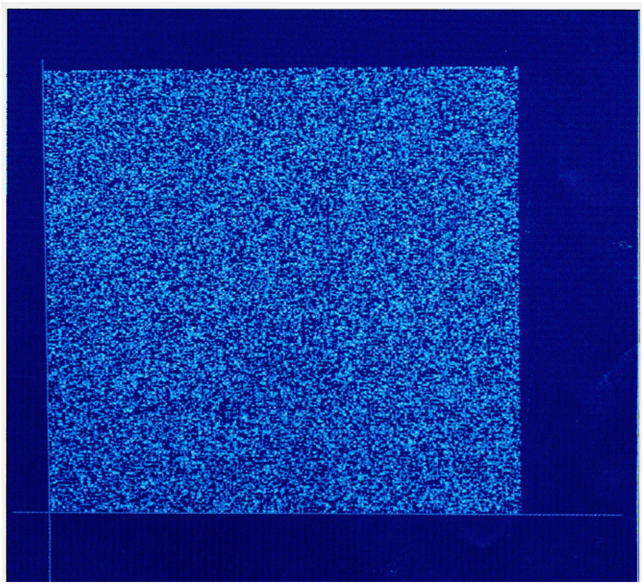

Fig. 2. In this figure, the map of Fig. 1 has been composed with a map which reflects the decimal positions of a number about the middle decimal value. In this form, the chaotic nature of the original map T is clear, but the reason for which it is chaotic is obscured by its high-dimensional domain of definition.

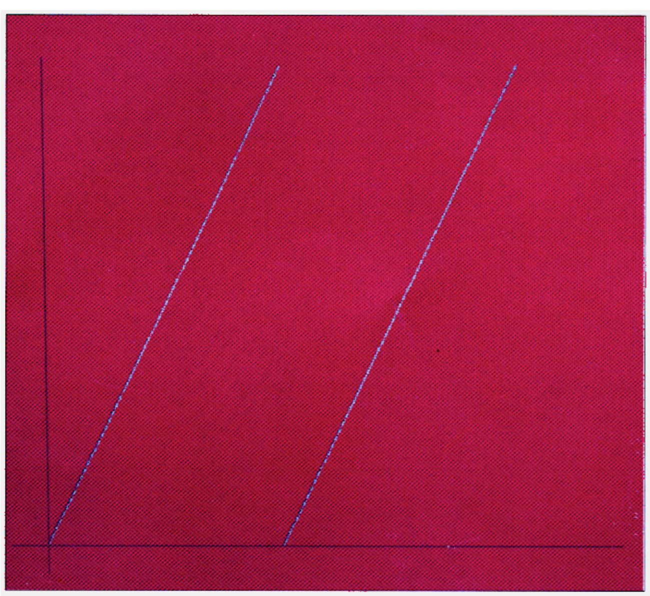

Fig. 3. In this figure we show that the simple coordinate shift generates $2x \bmod (1)$. The orbits of this map are short in comparison with those of Figs. 1 and 2, while both maps are essentially chaotic.

A simple coordinate shift produces $2x \bmod (1)$, i.e. $x_i \to x_{i-1}$. The graph determined by this conjugacy is shown in Fig. 3.

Plotting this figure requires that the dimension of the domain of T is large enough to get a good graph. A dimension greater than 1000 will suffice. By iterating T when defined to be conjugate to $2x \bmod (1)$, the orbits cannot exceed the dimension of the domain space. This is not true of $3x \bmod (1)$ even though both $2x \bmod (1)$ and $3x \bmod (1)$ are shifts on the appropriate symbol set.

3. Extending T to Finite Invariant Subsets with Integer Coordinates

The invertible mapping π used to obtain a conjugacy between T and the one-dimensional map S_{T} can be defined for any T-invariant set whose coordinates are integers. The essential step in doing this is keeping track of multiple digit integers. For example, if the point $(21, 3)$ is mapped to 0.213, it cannot be inverted without more information since the point $(2, 13)$ also has this representation. The solution of this invertibility problem is the choice of a useful radix. If 21 is the largest integer to appear in a coordinate of a finite subset of the plane, then the choice of the radix 22 will assure that we can define π so that it is invertible. To continue this example, suppose our radix is 22, then 10 corresponds to 22, and the numbers below 22 are labeled as $0, 1, 2, \ldots, 9, a_{10}, a_{11}, a_{12}, \ldots, a_{21}$. In this system the point $(21, 3)$ is $(a_{21}, 3)$ and the point $(2, 13)$ is $(2, a_{13})$. This removes the ambiguity since $\pi((a_{21}, 3)) = 0.a_{21}3$ and $\pi((2, a_{13})) = 0.2a_{13}$. These numbers can be converted to decimal using the usual formulae, thus assuring that we can see the dynamics of the one-dimensional map in the usual way. Given this we have the result:

Let T be a mapping of $\mathbf{R}^n$ into itself which preserves a finite subset, $\mathcal{V}_k^n$, whose coordinates are integers less than or equal to k. Then T is conjugate to a one-dimensional map when restricted to $\mathcal{V}_k^n$. This follows by using the radix $k + 1$ to define π, and then defining $S_{\mathrm{T}} = \pi \circ \mathrm{T} \circ \pi^{-1}$.

4. Dynamics of Boolean Automata

Boolean automata are automata whose states are either 0 or 1. Typically, these automata are modeled using Boolean logic. In this section, we show that these automata can be viewed as dynamical systems on high-dimensional manifolds.

It is sufficient to show that every Boolean expression has an algebraic expression which is equal to the Boolean expression when restricted to the set $\{0, 1\}$. We begin with binary expressions. The Boolean expression for *and* is given by $x \wedge y$. The algebraic equivalent is $x \cdot y$. The expression for Boolean *or* is $x \vee y$, whose algebraic equivalent is $x + y - x \cdot y$. This could be derived by using the Boolean expression for the complement of x, whose algebraic equivalent is $1 - x$. *Exclusive or*, $\underline{\vee}$, is not derivable from *and* and complements, and must be separately derived. Its algebraic expression is $x + y - 2 \cdot x \cdot y$. *If x then y* is given by $\sim x \vee y$, which is $(1 - x) + y - (1 - x) \cdot y$, which simplifies to $1 - x + x \cdot y$. From these elementary formulae the algebraic expression for any Boolean expression can be deduced, thus making it possible to use elementary algebra to reduce a Boolean logic expression to its simplest form. For example, $(x \wedge y)\underline{\vee}z$ is given by $x \cdot y + z - 2x \cdot y \cdot z$. The algebraic formulae for Boolean logic makes it clear that Boolean expressions produce nonlinear dynamics, and that finite automata are generally nonlinear dynamical systems. Since finite automata can be examined through one-dimensional dynamics, this raises the question "are there simple Boolean formulae which give rise to complex one-dimensional dynamics?". Our construction of the mapping $3x \bmod (1)$ is an example of a finite automata which has chaotic dynamics. But the formulae for this map is relatively complicated due to the carry function. A simpler formula with complex dynamics is given by

$$x_i \to x_i \cdot x_{i-3} + x_{i+3} - 2x_i \cdot x_{i+3} \cdot x_{i-3}$$

where $0 \to M$, $-1 \to M-1$, $-2 \to M-2$, and, $i+3$ is $\bmod(M+1)+1$, with $M = 81$, the dimension of the manifold.

Figure 4 shows the one-dimensional attractor for this system. In this system the value of a cell is a function of its value and the value of the cells three steps before and three steps afterward. If this system were laid out as a 9 by 9 array of cells, it would happen that the value of a boundary cell such as cell 10, the first cell in the second row, is modified by cell 7 and cell 13. This asymmetry is a curious consequence of viewing a dynamical system as a cellular automata laid out on a two-dimensional plane (if it is laid out as a torus, this phenomena

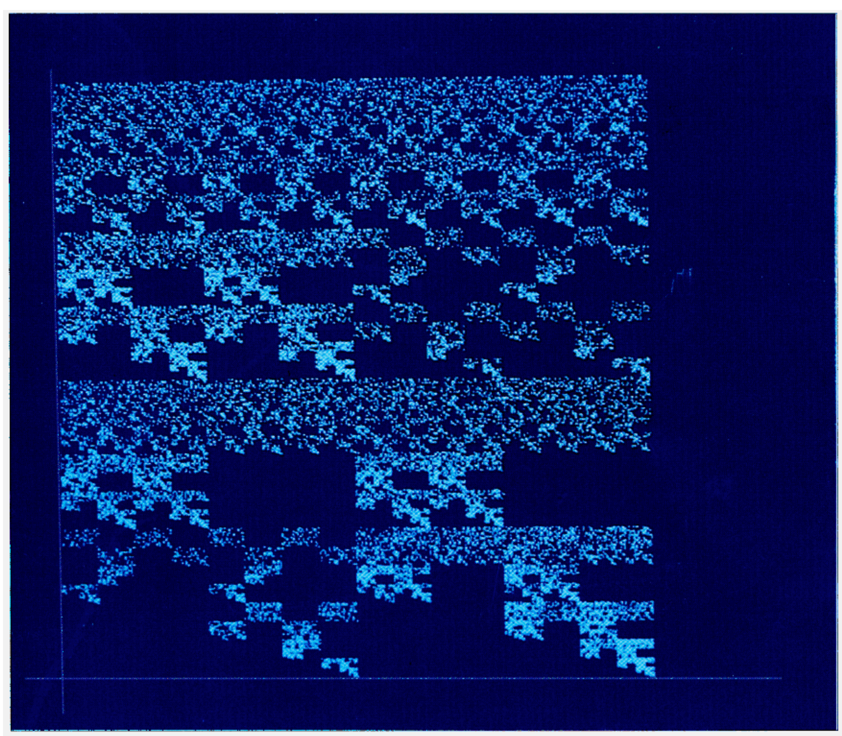

Fig. 4. In this figure we see that very complicated orbits can be generated without resorting to the use of known chaotic maps such as $3x \bmod (1)$. While observing that shift and add is sufficient to generate chaos in Figs. 1 and 2, no such observation will suffice to explain Fig. 4.

goes away). More interesting is that any automata that is defined to have each cell only modified by its neighbors translates into a dynamical system with asymmetries in its definition. This increases the difficulty of writing down a simple formula for the dynamical system. Still more interesting, the consequence of a rule whereby a cell's value is only affected by that of its neighbors translates to having a digit in a binary number affected by digits far removed from it. Thus, significant digits in a number can be affected, or altered, by the value of insignificant digits. This fact provides an insight into how complex dynamics can evolve unexpectedly, since having an insignificant digit in a number affect the value of significant digit means that the meaning of the term *significant digit* is altered.

An example of this is having the digits in a number reflected about a central number. For example, if $x = 0.123456789$ then map x to the number 0.987654321. This is not easily done in a formula for a one-dimensional map, but it is very easy for a high-dimensional mapping. The transformation, T, is given by $x_i \to x_{N-i+1}$ where N is the dimension of the space. In effect, this mapping scrambles the significance of the digits represented by the mapping, T, restricted to points whose coordinate entries are in the set $\{0, 1\}$. In 1988, Lin and Chua discovered a version of this type of chaos, which might be called discrete chaos, in digital filters, see [Chua & Lin, 1988; Lin & Chua, 1991]. A symbolic dynamic structure for their findings was developed by Wu and Chua [1994]. Ogorzalek examined the presence of complex behavior in digital filters noting that both chaotic and nonchaotic complex behaviors can be observed and examined using one-dimensional maps, see [Ogorzalek, 1992]. Our presentation provides a formal framework that explains their results. Specifically, complex dynamics can be observed in digital filters due to the conjugacy of their implicit high-dimensional maps to chaotic and complex one-dimensional maps similar to those given in our examples.

As mentioned, in order to generate complexity, it is not necessary to restrict our attention to chaos. For example, it is possible to carry out the same constructions using skew translations or mappings which are weak mixing. The result is a high-dimensional dynamical system that maps $\mathcal{V}^n$ into itself in which the presence of the skew translation or weak mixing transformation is concealed by the large number of dimensions.

5. Conway Dynamics

In this section we derive a dynamical system for John Conway's *Game of Life*. In this regard, we mention that Dogaru and Chua [1999] derived the simplest CNN realization of the Conway Game of Life. Their analysis realizes the Game of Life as a simple explicit formula involving only absolute value functions. In this analysis we derive, by an alternate approach, a dynamical system whose restriction to lattice $\mathcal{V}^n$ is the Game of Life.

Conway's cellular automata is a $M \times M$ array of cells each having the value 0 or 1. The rules for this automata are as follows:

(1) If a cell has the value 1, and has either 0, 1, 4, 5, 6, 7, or 8 neighbor cells having the value 1, the cell's value changes from 1 to 0. (2) If the cell has two or three neighbor cells with the value 1, its value remains 1. (3) If the cell has the value 0, and there are three neighbors with the value 1, its value changes from 0 to 1.

It is possible to derive the corresponding dynamical system utilizing the Boolean Dynamics of Sec. 4. However, this approach would require over 100 terms in the coordinate transformation defining the Game of Life dynamics. A much shorter version can be obtained by utilizing a technique from [Brown & Chua, 1993]. With this approach, the ith coordinate can be expressed as

$$x_i \to x_i v_1 + (1 - x_i) v_2$$

where $v_1 = 0.5(1 + \mathrm{sgn}((y_i - a_{11})(a_{12} - y_i) + 0.05))$, $v_2 = 0.5(1 + \mathrm{sgn}((y_i - a_{21})(a_{22} - y_i) + 0.05))$, and

$$y_i = \sum_{j=1}^{8} x_{f_i(j)}$$

The function, $f_i(j)$ is given by $f_i(1) = (i + 1)$, $f_i(2) = i - 1$, $f_i(3) = i - M - 1$, $f_i(5) = i - M + 1$, $f_i(6) = i + M - 1$, $f_i(7) = (i + M)$, $f_i(8) = i + M + 1$. f is defined only for the integers one to eight since Conway's definition utilized only eight cells of the cellular automata. For the Game of Life, we must set $a_{11} = 1.5$, $a_{12} = 3.0$, $a_{21} = 2.5$, $a_{22} = 3.0$.

Some care must be taken to ensure the boundary cells function properly, but this is not a problem. The value of this formulation is the ability to easily manipulate the parameters a_{ij} appearing in the dynamical systems formulation. If we change a_{21} from 2.5 to 2.0, the dynamics change drastically. Whereas typically, the Conway parameters

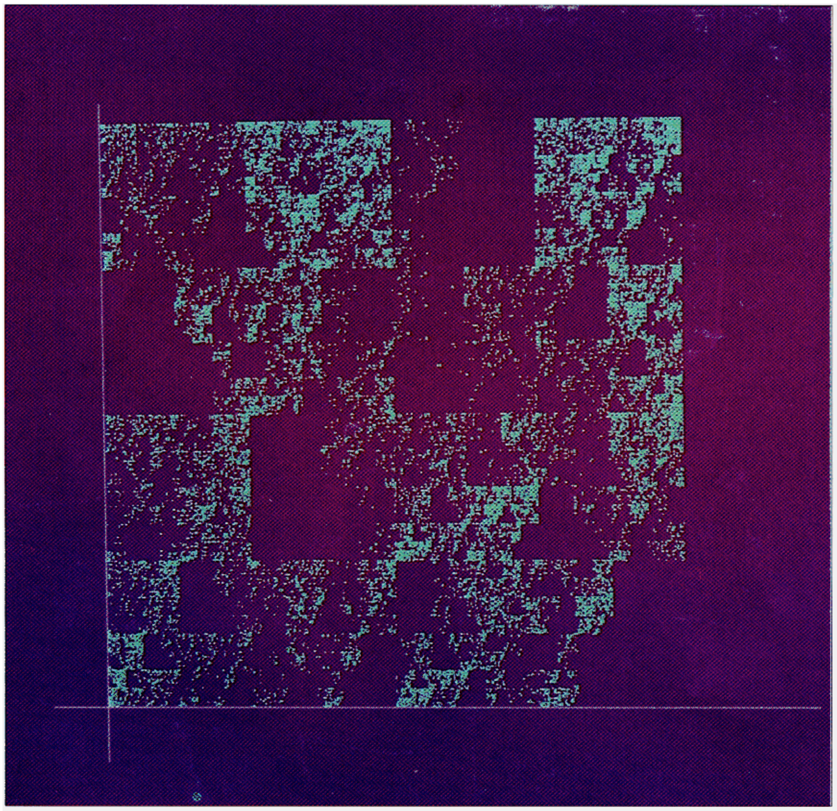

Fig. 5. In this figure we observe that if the equations for the *Game of Life* are slightly modified, the orbits become chaotic. Thus the dynamical systems formulation of this CA provides a useful tool for studying variations on the Conway's interesting paradigm.

lead to attracting fixed points very quickly, this modification typically leads to very long transient chaotic orbits of thousands of points. Figure 5 is the one-dimensional map for a typical modified Conway orbit.

Using the methods in [Brown & Chua, 1993] this mapping may be realized as the time-one map of an electronic circuit.

6. Stabilizing T

In general, $\mathcal{V}^n$ is an unstable set when T is defined using Boolean functions of Sec. 4. As noted, if values slightly off $\mathcal{V}^n$ are used as initial conditions, the orbit of the transformation will either rapidly converge to the origin, or go unbounded. However, it is possible to define a new transformation that agrees with T on $\mathcal{V}^n$ for which $\mathcal{V}^n$ is a stable set. This is done as follows: Let T be a transformation defined as

$$\mathrm{T}(X) = \begin{pmatrix} f_1(X) \\ \vdots \\ f_n(X) \end{pmatrix}$$

where $X \in \mathcal{V}^n$ and $f_i(X) \in \{0, 1\}$ for each i, and each f_i is a first-order polynomial. Then the following transformation agrees with T on $\mathcal{V}^n$ and is stable.

We first define an auxiliary function that is useful

$$\mathrm{S}(X) = \sum_{i=0}^{n} x_i^2 \cdot (1 - x_i)^2$$

where $X = (x_1, x_2, x_3, \ldots, x_n)$. Using this function we now define a new map that agrees with T on $\mathcal{V}^n$ and is stable. This transformation is

$$X \to \exp(-(\mathrm{S}(X)))T(X)$$

In addition to stabilizing the dynamical system, we can make it differentiable by substituting the hyperbolic tangent for the sgn function, as done in [Brown & Chua, 1993].

7. Summary

The subject of complexity can be divided into two parts. One part is the study of complex systems which cannot be investigated with the use of mathematics. The second part is the study of complex systems which can be investigated through mathematics. It is only the second part that we address. In this regard, we are interested in complex systems whose "scheme" can be described in objective terms and can be resolved into formulae which can be manipulated to extract insights, make predictions, or construct replicas and accurate models.

It is now generally thought that what is mathematically complex lies between that which is simple, such as periodic dynamics, and that which is pseudorandom, such as Bernoulli dynamics. The reason for this is that it is easy to write down examples of periodic dynamics and of Bernoulli dynamics, but it is hard to write down examples of things in between such as weak mixing dynamical systems which are not strong mixing. In this sense, a transformation that is weak mixing but not strong mixing is more complex than a Bernoulli mapping, even though its mixing properties are far weaker.

Difficulty in writing down a formula is not enough for a system to be complex. It must also be hard to figure out, or predict, from measurements. For instance, there are easy examples of almost periodic motion which are hard to write down, but they are still easy to predict from measurements. It is the combination of being hard to write down and hard to predict from measurements that makes something complex. A weak mixing transformation which is not strong mixing is both hard to write down and hard to predict from measurements. Generally, any system having sensitive dependence on initial conditions (it does not have to be exponentially sensitive as with chaos) which cannot be captured in a formula is complex, since sensitive dependence on initial conditions makes the use of measurements for prediction problematic.

The complexity in stating a dynamical system as a formula (which may be an ODE, PDE, finite difference equation, or a replication equation) may be related to dimensionality. We state the following conjecture in this regard:

Conjecture. *Given any one-dimensional map having only a countable number of discontinuities, there exist a realization of this map as a projection onto the unit interval of a smooth dynamical system of sufficiently high dimension.*

The spirit of this conjecture is that complexity in the formulation of a low-dimensional dynamical system can be traded off for an increase in the dimensionality of the system. Thus, in order to mathematically describe complex dynamics, we can be forced to formulate the system in a high-dimensional space. When these systems are projected onto one dimension, there is a loss of the desirable property of differentiability, and even continuity. In fact, total discontinuity may result from our efforts to visualize complex systems in low-dimensional spaces. To preserve differentiability, as the examples in this paper show, we may often have to increase the dimension of the domain of the system.

The view of a complex system as a smooth transformation on a high-dimensional space affords insight through having a formula to examine. Conversely, projecting a complex system onto the unit interval provides a graphic with which to examine and compare complex systems visually. Both views provide a window into complex dynamics where we can investigate how complex dynamics arises from a large number of simple systems interacting to produce complicated space-time orbits, and whose interaction can be very involved to express, even in many dimensions.

Acknowledgments

We would like to thank Jim Crutchfield and Andrew Wuensche of the Santa Fe Institute for valuable conversations about the contents of this paper.

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