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CHAOS: GENERATING COMPLEXITY FROM SIMPLICITY

RAY BROWN
Applied Chaos Technology Corporation,
3865 Wilson Blvd. Suite 210, Arlington, VA 22203, USA

LEON O. CHUA
Department of Electrical Engineering and Computer Science,
University of California, Berkeley, CA 95720, USA

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The most commonly used mapping to illustrate the phenomenon of chaos is the map $x \to 2x \bmod(1)$. This map is known as the 'unilateral shift' because, in the binary number system this map shifts all digits to the left by one decimal place, and truncates the integer. The second most commonly used paradigm of chaos is the Smale horseshoe whose complexity is essentially the bilateral shift obtained when we simply shift without truncation in some symbol system. Neither of these paradigms fully explains chaos since shifts cannot generate complex orbits from simple (rational) initial conditions. How chaos generates complexity from simplicity is an essential part that needs explanation. Providing this explanation is the objective of this paper.

1. Introduction

In this paper we bring attention to an important feature of chaos unexplained by the shift, the Smale horseshoe, or related paradigms. In physical terms, this feature is the ability of a dynamical system to produce very complex outputs from simple initial conditions and parameters. In mathematical terms this feature is the ability of an algorithm to generate an irrational number from rational initial conditions and parameters. An example is the classical algorithm which produces $\sqrt{2}$. We call such dynamical systems *demiurgic*. The significance of this class is that these algorithms give us direct insight into how a chaotic dynamical system can generate complexity from simplicity.

In Sec. 2 we cover some preliminary ideas needed for our exposition. In Sec. 3, we present examples that illustrate a property of dynamical systems that we call demiurgic. In particular, dynamical systems may be divided into demiurgic and nondemiurgic systems. The shifts are nondemiurgic systems, as we will see. We will also show that commonly encountered chaotic dynamical systems are often demiurgic and are thus unlike the shift in that they can generate complexity from rational initial conditions. In Sec. 4, we show how unbounded dynamical systems such as $\dot{x} = x$ are connected to chaos and demiurgic systems. Section 5 is a summary.

2. The Shift, the Complexity Spectrum, and Attracting Fixed Points

2.1. *The shift paradigm*

Paradigms of chaos are intended to convey intuitively how very complex behavior can arise in nature. In this regard, the most long-standing mathematically rigorous paradigm for chaos is the Smale horseshoe. The intent of this concept is to demonstrate how stretching and folding can lead to the highest level of complexity known to exist in dynamics. Crucial to the argument is to show how the complexity that results from stretching and folding in a dynamical system can be realized in a map called the bilateral shift. Smale's theorem specifically shows that under broad conditions a mapping can have the same dynamics as a bilateral shift on some invariant subset of its domain. We are thus led to infer that when this occurs, the dynamics of the map of interest on this invariant set, often having measure zero, extends in some unspecified way to the entire domain or at least to a domain of positive measure, and thus there arises the phenomenon of chaos.

Smale's theorem has to a large degree shaped our thoughts about chaos and, consequently, we have come to consider the bilateral and unilateral shift the two most universal examples of chaotic dynamics.

2.1.1. *The source of chaos as explained by the shift paradigm*

Due to the historical importance of Smale's theorem we review the connection of the bilateral and unilateral shift to chaos. In this regard, one of the most persuasive presentations of this theory was given by Ford [Barnsley & Demko, 1986]. His argument, for simplicity, uses the unilateral shift

$$x \to 2x \bmod(1)\,.$$

He argues that the source of chaos in the shift is deduced from the observation that if the initial condition x_0 is a very complicated number, then this level of 'complexity'[1] will be reflected in the orbit. The argument depends on using the binary representation of numbers. Thus if

$$x_0 = (.1011000111001011010\ldots)$$

then, in binary

$$2x_0 = (1.011000111001011010\ldots)$$

and applying the mod(1) function gives the number

$$2x_0 \bmod(1) = (.011000111001011010\ldots)$$

Clearly, all orbits of this map are determined solely by their initial conditions. If the initial condition is an infinite 'complicated' series of digits, then the orbit will be also. It is a very significant point that, as a result, the orbit of a shift does not contain any digit sequence not already present in the initial condition. This line of thought reduces the explanation of chaos to the question of "*How 'complicated' can an initial condition be*?" This problem was addressed by Kolomogrov, Chaitin and others and led to the development of an important and interesting mathematical discipline, see [Brown & Chua, 1996a]. There are only two details from that discipline that are essential for our argument here: (1) A binary number has the highest level of 'complexity' when the only way it can be communicated is to write the number down long-hand, i.e. there is no finite algorithm that we can use to generate the number on a computer; (2) From the point of view of measure theory, most numbers are of this form. What is important here is that the explanation for the phenomena of chaos is that chaos results — in a class of dynamical systems that have yet to be satisfactorily specified — from the uncertainty inherent in our inability to measure the *initial conditions and parameters* in a physical system.

For the shift, if there is any measurement error in the initial conditions, this translates directly into errors between the predicted and actual orbits. Since an initial condition in binary can be generated, in theory, by a coin toss, heads = 1, tails = 0, then the orbit that results from this initial condition is essentially as random as a coin toss. Thus the shift illustrates the properties of both a deterministic and a random process, i.e. it is described by a formula, yet an orbit of the shift can be as random as a coin toss. The paradigm of deterministic-randomness arising from the initial conditions is at the heart of the shift, unilateral or bilateral, and is the *point* in proving the existence of a horseshoe.

Clearly, the initial conditions can be a factor in the appearance of chaos and this is explained by the shift maps. But the other side of the coin is that a chaotic dynamical system can start from very simple initial conditions, rational numbers which may be integers, and generate very complex, unpredictable orbits. This can not be explained by the shift.

[1] A complete explanation with references of what is rigorously meant by the word *complexity* is found in [Brown & Chua, 1996a].

The proof that a given system behaves like a shift on some subset of its domain turns out to be so difficult that alternative definitions of chaos have evolved, see [Brown & Chua, 1996a]. The emergence of the positive Lyapunov exponent as a definition of chaos was one response to the mathematical difficulties in proving the existence of horseshoes. This definition was based on the property of the shift that caused the complexity of the initial conditions to be reflected in the orbit. This property originates from the number 2 in the mapping $x \to 2x \bmod(1)$. In other words, the derivative of the shift has an eigenvalue whose absolute value is greater than 1. A map or a flow has a positive Lyapunov exponent on an orbit when, on the average, the derivative has an eignevalue whose absolute value is greater than 1. This popular definition is not theoretically difficult to apply, but can be numerically and computationally a nightmare to verify. But the thinking behind this definition is that chaos arises from the initial conditions, and that the positive Lyapunov exponent operated just like the shift to make the complexity of the initial conditions appear in the orbit. (In Example 5, we will show that having a positive Lyapunov exponent is not the same as being a shift in that it *can* sometimes produce a complex orbit from simple initial conditions.) Other definitions have virtually canonized the idea of sensitive dependence on initial conditions as being equivalent to chaos. It is conceivable that sensitive dependence on initial conditions might be the only universally accepted definition of chaos. However, the examples in [Brown & Chua, 1996a] show that even simple nonchaotic system can have this property.

In short, paradigms of chaos are focused on the initial conditions as being the source of chaos rather than the algorithm, or dynamical system. In this section, we direct the focus from the initial conditions back to the dynamical system.

2.1.2. *Problems with the horseshoe/bilateral shift paradigm*

There are four fundamental problems that arise with using this paradigm as an explanation of chaos.

(1) The type of initial conditions that are needed for this paradigm to be used as a general explanation of chaos can never be constructed, measured, or verified in any physical observation or process.
(2) The dynamics of the shift can never *generate* a complex orbit from a rational initial condition.
(3) The dynamics of the bilateral shift, or horseshoe, usually occur on a set of measure zero, and thus may be physically *insignificant.*
(4) Maps which have a horseshoe are only *topologically* conjugate to the shift, not differentiablly conjugate, and hence are not related to the shift in a physically *smooth* way.

With regard to the first problem, one could argue via measure theory that the kind of initial conditions that produce complex orbits in the shift are the most common kind. That is, if we could put all numbers on the interval [0,1] in a bag and reach in and pull out one, it would be one of this type with probability 1. However, because of (3) this is of no importance.

Of all of these problems, (2) is the most troubling because it is contrary to our observations of, and our intuition about, the physical world. True complexity does arise in the physical world from simple beginnings. We can see it and measure it and reproduce the experiments of others and get their results. None of this can be explained by the shift. The shift only tells us that deterministic dynamics exist that are *virtually* random. But it cannot account for the *generation* of complexity from simple beginnings that we see, and it is the generation of complex things from simple noncomplex parts that is just as important a feature of chaos as the unpredictable evolution of chaotic orbits.

We conclude that the dynamics of the shift, while a partial explanation, cannot be a *complete* explanation of the phenomenon of chaos.

2.2. *Chaos and the spectrum of complexity*

In three recent papers we have presented over 50 examples, [Brown & Chua, 1996a, 1996b, 1998] to show that it is very difficult to formulate a definition of chaos that is universal. Serendipitously, these examples serve to illustrate that there exist a very broad spectrum of complexity in the universe. Within this spectrum, we see systems that range from constant, to periodic, to almost periodic, and on to chaotic, and then further on beyond chaotic to what may be called *gamma chaos* for its relation to gamma functions, see [Brown & Chua, 1996a]. Still further on are possibly systems that can be described by algorithms that are even more complex than gamma chaos. At some point in this spectrum we arrive at a system that cannot be described by any finite algorithm, so we call these systems *random* for want of a better word. What lies within this region of the spectrum of complexity is unknown and is perhaps unknowable.

The existence of this spectrum suggests that to understand chaos we must first answer the question "*How does chaos fit into the spectrum of complexity?*" or "*What set of attributes uniquely distinguishes chaos from the other forms of complexity?*". Secondly, of fundamental importance is the question "*Does chaos occupy a distinguished position in the universe, in biological systems, or in some realm of human endeavor?*." For example, is chaos the engine of life? Is chaos the engine of the solar systems?

At present, as examples that have been published or to appear have shown, we are not able to answer these questions. In particular, no definition, i.e. fractal dimension, entropy, Lyapunov exponents, are able to pinpoint the exact location of chaos within the spectrum of complexity and distinguish it from other forms of complexity that are found there.

2.3. *The complexity spectrum and attracting fixed points*

In this section we tie together systems that produce some level of complexity with those which have only attracting fixed points. This relates the complexity spectrum to symbol sequences.

The unpredictability of the integer sequence of the decimal portion of π is a level of complexity. In general, systems having attracting fixed points which are unpredictable irrational numbers determine a level of complexity.

Using the decimal as a marker for a starting point, we may view an irrational number as a sequence of integers between 0 and 9, as is done typically in symbolic dynamics. The setting is the set of all mappings from $\mathcal{M} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ into itself. With this set, a decimal number can be expressed as a function $f : \mathcal{M} \to \mathcal{M}$. Immediately there is a problem with this setting. Since there exist irrational numbers so complex that there is no finite algorithm that can be used to write them down, some functions in this set cannot actually be written down, or constructed. This forces us to organize our thinking about these functions into two classes. The functions that can be written down will be called *deterministic* and those that cannot will be called *random*. This imposes a natural division of irrational numbers into deterministic and random that closely parallels our ideas about random sequences such as those generated by a coin toss. Further divisions are possible: Periodic functions correspond to rational numbers which are periodic. Rational numbers which are eventually constant have two descriptions. They may be called converging sequences or they can be called functions which are eventually periodic. The division of almost periodic makes sense in this setting. After this division we have an array of numbers, functions, or sequences that are more complex than almost periodic but are less than random. These functions are characterized by increasing levels of complexity.

Within this setting, how do we construct the function that represents the sequence of digits in the square-root of 2? To illustrate the difficulties of this construction we note that the function that corresponds to the number $.1234567891011121314\ldots$ is fairly involved, see Example 6. If we can find any algorithm to generate $\sqrt{2}$, then by a construction like that of Example 6 we can generate the associated f. This fact frees us to restrict our attention to algorithms such as Newton-Raphson, taking on faith that we can fill in the missing function gap if we are determined to do so.

Starting with an initial condition of 1, each iteration of Newton–Raphson produces a better approximation to $\sqrt{2}$. Each approximation defines a function on $\mathcal{M}$. At the end of Newton–Raphson, which can never actually be reached, is $\sqrt{2}$.

Can we classify, within the complexity spectrum, the function defined by $\sqrt{2}$? For example, suppose that we have a bounded dynamical system defined on a space. Following symbolic dynamics, we partition the space into 10 regions, labeled 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. Now, we take an initial condition in region 1 and iterate the dynamical system, i.e. generate the orbit. If the symbol sequence generated by this orbit is $\sqrt{2}$, is the system chaotic, or is it some other category of complexity?

We may define the dynamical complexity of a symbol sequence by the level of complexity of the simplest dynamical system that can be used to generate the sequence. In this way we connect the type of dynamical complexity to the complexity of symbol sequences and through this we connect dynamics to the construction of irrational numbers from integers.

3. Generating Complexity and Chaos

In this section we argue that while complexity in the initial conditions of a system *can* give rise to a level of orbit complexity, there is another mechanism that is independent from this that gives rise to a level of complexity regardless of the nature of the initial conditions. In order to see what is missing from the shift we take a close look at the various features of the shift that account for its ability to produce complex orbits.

The shift has two key properties:

(1) A mechanism to move lower level digits into higher level positions of significance, i.e. multiplication by the integer 2.
(2) A mechanism of eliminating higher level digits, i.e. the mod function that eliminates the integer part of a number.

What we do not see in any shift is:

(3) A mechanism of adding digits onto the decimal part of a rational initial condition.

Each of these three mechanisms is independent of the other two. The first mechanism is illustrated by the multiplication of an irrational number such as π by n, n^2, 2^n, etc. For example, if the decimal part of π, i.e. $0.14159\ldots$, is taken as an initial condition then multiplication by 10 gives $1.4159\ldots$. Discarding the integer portion we get $0.4159\ldots$. Doing this three more times moves the number $0.9\ldots$ into a position in which it could be significant for measurements purposes. That is, after four iterations, 9 is now in the first decimal position, whereas it started at the 5th decimal position. This is what dynamical systems whose derivatives have eigenvalues whose absolute value is greater than 1 do to the initial conditions. A system having a positive Lyapunov exponent may move a given decimal position of an initial condition up and down in significance, but on the average it moves it up.

The second mechanism is illustrated by periodic functions of which $f(x) = x \bmod(1)$ is an example. Because of the periodicity, the significance of the integer portion of a number diminishes after each period. This is an entirely different dynamic from (1).

While the maps that move a given decimal position upward into significance for measurement purposes are important to understanding chaos, equally important is the class of maps which are capable of adding on decimal digits to rational initial conditions, or which are capable of changing the decimal portion of rational numbers to make them irrational. Examples of the latter class are the algorithms that generate $\sqrt{n}$, (n is prime), π, γ, (Eulers constant), e, and other classical constants.

An interesting example is suggested by DNA structures. Consider the human gene for β-globin which symbolically starts CCCTGTGGAGCCACA... and goes on for about 2000 more entries.[2] The generation of this gene can be conceived as taking place through the action of a one-dimensional dynamical system having an attracting fixed point which is a decimal fraction whose first 2000 or so decimals values coincide with a symbolic dynamic coding of the gene. For example, symbolic dynamics suggests that we set A = 1, C = 2, G = 3, T = 4 to get the number $0.222434331322121\ldots$ If we ask what dynamical system could generate this number we are also indirectly asking "What dynamical system could produce this gene?".

A dynamical system that could generate gene codes in this manner might be called *demiurgic*. The phenomenon of spontaneous polymerization of the four basic nucleotides with the loss of water is an example of what we mean by demiurgic, see [Alberts *et al.*, 1989, p. 5, Figs. 1–3]. Thus, for want of a better word, we will say that a map is *demiurgic* if it has mechanism (3). Using this terminology we can say that not all demiurgic maps are chaotic as demonstrated by the square-root algorithm. Not all chaotic maps are demiurgic as illustrated by the shift. Not all demiurgic maps have attracting fixed point. But some chaotic maps are demiurgic, perhaps most, and perhaps this is the most important class of chaotic maps.

We now proceed to a series of examples designed to clarify the role of the demiurgic feature of a map in producing complexity.

4. Examples

We use the square root algorithm as the basis of our examples due to its broad familiarity, and its role in the subject of fractals. Using Newton–Raphson

[2]A, C, G, and T are abbreviations for the four fundamental nucleotides that make up all DNA, see [Alberts *et al.*, 1989].

iteration to solve the equation $x^2 - a = 0$ leads us to the algorithm:

$$x_{n+1} = (x_n^2 + a)/(2x_n)\,.$$

As a dynamical system this is

$$x \to (x^2 + a)/(2x)\,.$$

For the special case of the initial condition being a rational number p/q, with p, q integers, we get the algorithm

$$\begin{pmatrix} p_{n+1} \\ q_{n+1} \end{pmatrix} = \begin{pmatrix} (p_n^2 + aq_n^2) \\ 2p_n q_n \end{pmatrix}$$

which defines a dynamical system on the two-dimensional lattice of points having integer coordinates. It is significant that this system is unbounded on the integer lattice, even though the ratio, p/q, converges to a fixed point.

Example 1. The Basic Example. In this example, we take one coordinate to generate the square root of a prime number. We use a second coordinate to generate the sequence of integers, and use a third coordinate to multiply these two and remove the integer part. The algorithm looks like this:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} (x^2 + a)/(2x) \\ y/(y+1) \\ (xf(1/y)) \bmod (1) \end{pmatrix}.$$

The x-coordinate converges to $\sqrt{a}$ and is the algorithm that one obtains by a direct application of the Newton–Raphson method to find the root of the equation $x^2 - a = 0$. The second coordinate is derived as follows: If

$$u_{n+1} = 1/(n+1)\,,$$

then

$$u_{n+1} = 1/((1/u_n) + 1) = u_n/(u_n + 1)\,.$$

In the third coordinate we use the relationship $1/u_n = n$. Thus any function of $1/y$ is a function of n, so long as the initial condition for the y-coordinate is 1. Hence if $f(u) = u$, then at the nth iteration $f(1/y) = n$, and the product of x and $f(1/y)$ is nx_n at this step. Note that x_n is the nth approximation to $\sqrt{a}$ arising from iterating the first coordinate. Evaluating this product mod(1) discards the integer part and retains the fraction. If $f(u) = 2^{1/u}$, the third coordinate is $2^n x_n \bmod (1)$.

This simple algorithm generates complexity by producing the digits of an irrational number in the first coordinate. In the third coordinate we use the function f to move these digits into a more significant position, i.e. the first decimal place. We then make this decimal place significant by the use of the mod(1) function, which is the same as using the number as the argument in a periodic function. By choosing $f(n) = n$ we move the digits of the square root up very slowly. By choosing $f(n) = n^2$ we move them up very fast and the orbit becomes very 'random' looking. In fact, orbits may be uncorrelated. By using the 'binary-shift', $f(n) = 2^n$ in this case, to move the digits up we impart to the orbit the level of complexity commonly seen in chaos. A natural consequence of our construction is that the dynamical system has an attractor in a lower dimensional space. In this case the attractor is in a one-dimensional space. This example does not have a positive Lyapunov exponent or a horseshoe but is chosen as the prototype illustration of our concept for its simplicity. In the following examples we will show how to add positive Lyapunov exponents. But first we give a very important example to show that having a horseshoe or a positive Lyapunov exponent is not necessary to obtain the level of complexity of the shift.

Example 2. A map with all orbits of a unilateral shift without a positive Lyapunov exponent.

Let

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} x \\ y/2 \\ (z + x/y) \bmod (1) \end{pmatrix}$$

with the following initial conditions: $x_0 = a/2$, $a \in [0, 1]$, $y_0 = 1$, $z_0 = a/2$. The iterates of this map can be obtained in closed form as

$$\begin{pmatrix} a/2 \\ 1/2^n \\ (2^n a) \bmod (1) \end{pmatrix}.$$

Thus the z-coordinate has the same orbits as the unilateral shift but the eigenvalues of the derivative of this map are 1, 1/2, 1. There is no positive Lyapunov exponent. We may give this map a positive Lyapunov exponent with the following modification:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} x \\ y/2 \\ (\alpha z + x/y) \bmod (1) \end{pmatrix}$$

where $\alpha > 1$. The eigenvalues are then 1, 1/2, α.

Example 3. Generating complexity from rational initial conditions

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} f(x) \\ (\alpha y + x) \bmod (1) \end{pmatrix}$$

where f is any function defining, iteratively, any irrational number that can be defined by an algorithm from a rational initial condition. This means that the first coordinate converges to a fixed point, which is, therefore, an attracting fixed point. If $\alpha = 1$, then the complexity of digits of the fixed point are elevated into the first decimal by the multiplication by n. If $\alpha > 1$, the map has a positive Lyapunov exponent which is elevating the digits of the fixed point more rapidly. The orbits are as complex as the digits of any irrational number than can be generated by an algorithm.

Example 4. A modified skew translation which generates complexity.

Let

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} f(x) \\ (y + x) \bmod (1) \\ (z + y) \bmod (1) \end{pmatrix}$$

In this example, the eigenvalues are $f'(x), 1, 1$. Since f defines an attracting fixed point, there are no positive Lyapunov Exponents. The digits of the fixed point are being elevated into significance in the z-coordinate at the rate of n^2. Note that, strictly speaking, this is not a skew translation.

Example 5. Maps with Positive Lyapunov Exponents can be Demiurgic.

As mentioned above, the map

$$x \to 2x \bmod (1)$$

cannot generate any complexity. However, the map

$$x \to 1.9x \bmod (1)$$

can create complexity. The reason is that

$$1.9 = \frac{19}{10}$$

and thus multiplying by 1.9 is a shift to the left by the multiplication by 19, followed by a shift to the right by the division by 10. This situation is the composition of two different shifts, i.e. the shifts are in two unrelated bases.

As suggested at the start of this section, our ultimate interest is with mappings defined by rational parameters and initial conditions. We examine the above map more closely in this light. As noted it can be written as:

$$x \to (19/10)x \bmod (1)\,.$$

It is a special case of

$$x \to (p/q)x \bmod (1)$$

where p, q are integers having no common divisors. There is no general closed form solution for this equation even though for $q = 1$ there is a closed form solution. For $q > 1$, and p, q having no common divisors, this map produces an infinite orbit for rational x_0 so long as the numerator of x_0 is relatively prime to q.

The proof of this statement is as follows. First we express the map restricted to rational initial conditions as a two-dimensional mapping of integers. Let $x_0 = r/s$, where r, s are integers.

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} q^{n+1}s \\ (py_n) \bmod (qx_n) \end{pmatrix}$$

The ratio y_n/x_n forms the orbit of our original equation. By inspection, $y_n = p^n r - Kq$, K an integer (note that y_n cannot be expressed as $= p^n r \bmod (q)$). If the orbit is periodic then $y_n/x_n = r/s$ for some n, or $p^n r - Kq = rq^n$ leading to the result that $p^n r = Mq$ for some integer M. As q does not divide either r or p, we have a contradiction.

Thus there are infinitely many rational initial conditions, in fact a dense set, which generate infinite orbits of rational points. This distinguishes the map from the shift in a meaningful way.

This translates into the following. The mapping

$$x \to \alpha x \bmod (1)$$

can generate complexity if $|\alpha| > 1$, α is rational, and α is not an integer. So in order to start with a simple map and a rational initial condition, x_0, and generate complexity in a way that the shift cannot, we choose $\alpha = |p/q| > 1$ as a ratio of two integers which are relatively prime and require that the numerator of x_0 not divide q.

This example illustrates how having a positive Lyapunov is not the same as being a shift. Further, we see that what is important is not the Lyapunov exponent, but the actual eigenvalue of the derivative. If it is not an integer and is greater in absolute value than 1, then the system has the ability to generate infinite nonperiodic orbits even when we are using only rational numbers as initial conditions and parameters.

This example suggests that it is possible to view the shift as a special case of a larger class of maps which include demiurgic maps. Using symbolic dynamics we start with two sets of symbol sequences, say binary and ternary sequences. We assume a mapping exists between them that is $1-1$ and onto. For example, the unit interval has both a binary and ternary representation and a natural mapping between these two representations. We use $\mathcal{S}_n$ to denote the set of bi-infinite sequences on n symbols. Let $f : \mathcal{S}_3 \to \mathcal{S}_2$ be the natural $1-1$ onto mapping between these two sets. Let s_i be the left shift on $\mathcal{S}_i$. In this terminology, Example 5 corresponds to the map $s_2^{-1} \circ f \circ s_3$. The extensions of this mathematical structure are numerous.

This example suggests a dynamical system on the integers which can be solved in closed form whose understanding may be at the heart of the demiurgic class of maps.

Example 5a.

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} q^n \\ (p^n)\,\mathrm{mod}(q^n) \end{pmatrix}$$

where $p > q$ are distinct primes. The bounded, infinite sequence of nonperiodic rationals is the ratio y_n/x_n. This example points to the relationship between distinct prime numbers as having a key role in the phenomenon of chaos.

Example 6. Normal Numbers. An irrational number is called normal if the digits of the number, and all combinations of digits occur with the relative frequency of a uniform distribution. For example, the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 0 occur with frequency 1/10, all pairs occur with frequency 1/100, etc.

Normal numbers can be generated by algorithms. The simplest normal number is obtained by writing down the natural numbers in sequence:

$$\mathcal{N} = .123456789101112131415161718192021222324\ldots$$

This number can be generated on a computer to any desired level of accuracy. Using this algorithm as a first coordinate in place of the square root algorithm in Examples 1–4 produces a dynamical system that creates a high level of complexity, continually. By shifting these digits to the left, i.e. multiply by 10, and remove the integer part, we may generate an orbit on a computer that illustrates the very highest degree of chaos which arises from an algorithm, independently of the level of complexity of the initial conditions, parameters, or the presence of horseshoes or positive Lyapunov exponents.

We now develop a dynamical system based on $\mathcal{N}$.

$$\mathcal{N} = \sum_{N=0}^{\infty} \sum_{k=10^N}^{10^{(N+1)}-1} H(k,\, N)$$

where

$$H(k,\, N) = \frac{k}{10^{h(k,N)}}$$

and

$$h(k,\, N) = (N+1)k - 9\sum_{i=0}^{N} i10^{N-i}$$

and

$$\sum_{i=0}^{i=N} i10^{N-i} = N(10^{N+1}-1)/9 - 10^N + [(N-1)10^N - 10(10^{N-1}-1)/9]/9\,.$$

Since $N = \mathrm{INT}(\log_{10}(k))$, where INT is the integer part, we may consider N as a function of k and write $N(k)$ in place of N to get

$$g(k) = \frac{k}{10^{f(k)}}$$

where

$$f(k) = (N(k)+1)k - 9\sum_{i=0}^{N(k)} i10^{N(k)-i}$$

then,

$$\mathcal{N} = \sum_{k=1}^{\infty} g(k)\,.$$

Using this formula we can write down a dynamical system that generates $\mathcal{N}$:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} x/(x+1) \\ g(1/x) \\ z+y \end{pmatrix}$$

We must choose the initial conditions as (1, 1, 0). $\mathcal{N}$ is generated in the last coordinate. By adding the following coordinate we obtain an orbit which is as random as the sequence of integers in $\mathcal{N}$:

$$w \to 10^{(1/x)} z \bmod(1)$$

In order to make this the Poincaré map of an ODE we may use the procedure in [Brown & Chua, 1996a]. If we were to do this we would get an equation in 8 dimensions. Chaos can exist in any number of dimensions, but in this example, we had to start with a much higher dimensional space to get chaos to appear in a given dimension. In this case, it has taken a 8-dimensional space to produce one dimension of chaos in an ODE, and 4 dimensions to produce chaos in one dimension for a Poincaré map.

5. Connecting the Very Large to the Very Small

Example 6 can be generalized. If we consider the dynamical system defined by the map $h(n) = n+1$ and write down the orbit of $h^k(0)$ as $0, h(0)$, $h^2(0)$, $h^3(0), \ldots$ then remove the commas and add a decimal to the left we get

$$\mathcal{N} = .h^0(0)h^1(0)h^2(0)\ldots$$

In other words, $\mathcal{N}$ is formed from the orbit of a dynamical system in a routine manner. This construction is available for any dynamical system which has an orbit of integers. Thus we can form the number $.1248163264128\ldots$ from the system $x \to 2x$ where $x_0 = 1$. Clearly, a large class of irrational numbers can be obtained from integer orbits of dynamical systems. In particular, every dynamical system having an integer orbit that goes to infinity defines an irrational number in the interval $[0, 1]$. While most (in the sense of measure theory) irrational numbers are not *constructable* by this method, a dense set of irrational numbers is constructable in this manner.

By an entirely different method we may construct an irrational number from a pair of unbounded integer sequences. In particular, if n_k, m_k, $n_k < m_k$ are two unbounded sequences of integers, then n_k/m_k is a sequence of rational numbers in the interval $[0, 1]$. Since any sequence can be realized as the orbit of a dynamical system, the ratio of two sequences is also realizable as the orbit of a dynamical system. Reversing this idea we may now view chaotic dynamical systems having rational parameters and initial condition as unbounded dynamical systems having integer initial conditions and parameters. We illustrate this idea with the logistic map.

Example 7. The Logistic Map. The map $x \to 4x(1-x)$, for rational initial conditions, may be written as two integer equations for p_n, q_n and then the ratio, p_n/q_n can be used to obtain the orbit of the logistic map having rational initial conditions. The equations are

$$\begin{pmatrix} p_{n+1} \\ q_{n+1} \end{pmatrix} = \begin{pmatrix} 4p_n(q_n - p_n) \\ q_n^2 \end{pmatrix}.$$

In this way we are able to see that the logistic map with rational initial conditions is equivalent to integer equations which become unbounded. Now the question of periodic orbits for rational equations is reduced to a question of divisors on integer sequences. This means that if the initial conditions of the above equation are $p_0 = p$, $q_0 = q$ where q, p are distinct primes, and that the orbit is periodic, then there are integers m, n for which $p_m q_n = p_n q_m$. We know that $q_k = q^{2^k}$ by inspection. By induction we derive a contradiction, concluding that this orbit cannot, in fact, be periodic.

It is also possible to derive a similar result for the Hénon map. The arguments proving the existence of nonperiodic rational orbits are more related to number theory than to differential equations.

6. Summary

The shift-horseshoe paradigm explains the level of complexity that the n-body problem can achieve and it also explains one way that orbits can become uncorrelated. It does not explain how something complex can be generated from something simple. On the other hand, we have constructed examples to show that chaotic maps can generate complexity from simple initial conditions. This leads to a division of chaotic maps into two classes, demiurgic and nondemiurgic. The nondemiurgic chaotic maps such as the shifts do not directly appear in physical phenomenon, but are rather embedded on a set of measure zero, usually. In this case, the shifts are not necessarily physically significant and the information they convey about chaos is very indirect. In particular, the presence of shifts does not distinguish chaos from the rest of the class of complexity generating dynamical systems. Of note is that the definition of chaos that uses a positive Lyapunov exponent fares better as a paradigm for chaos when the map is also demiurgic.

With regard to the demiurgic property, our examples serve to demonstrate that the *relationship* between two unbounded sequences of integers is an important issue for the study of chaos and, therefore, this *number-theoretic* line of inquiry is important for understanding how complex structures evolve from simple structures.

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