International Journal of Bifurcation and Chaos, Vol. 3, No. 5 (1993) 1235–1267

DYNAMICAL SYNTHESIS OF POINCARÉ MAPS

RAY BROWN
Department of Mathematics, Howard University,
Washington, D.C. 20059, USA

LEON O. CHUA
Department of Electrical Engineering and Computer Sciences,
University of California, Berkeley, CA. 94720, USA

Received January 10, 1993; Revised April 15, 1993

We present a theory of constructive Poincaré maps. The basis of our theory is the concept of irreducible nonlinear maps closely associated to concepts from Lie groups. Irreducible nonlinear maps are, heuristically, nonlinear maps which cannot be made simpler without removing the nonlinearity. A single irreducible map cannot produce chaos or any complex nonlinear effect. It can be implemented in an electronic circuit, and there are only a finite number of families of irreducible maps in any n-dimensional space. The composition of two or more irreducible maps can produce chaos and most of the maps studied today that produce chaos are compositions of two or more irreducible maps. The composition of a finite number of irreducible maps is called a completely reducible map and a map which can be approximated pointwise by completely reducible maps is called a reducible map. Poincaré maps from sinusoidally forced oscillators are the most familiar examples of reducible maps.

This theoretical framework provides an approach to the *construction* of "closed form" Poincaré maps having the properties of Poincaré maps of systems for which the Poincaré map cannot be obtained in closed form. In particular, we derive a three-dimensional ODE for which the Hénon map is the Poincaré map and show that there is no two-dimensional ODE which can be written down in closed form for which the Hénon map is the Poincaré map. We also show that the Chirikov (standard) map is a Poincaré map for a two-dimensional closed form ODE. As a result of our theory, these differential equations can be mapped into electronic circuits, thereby associating them with real world physical systems. In order to clarify our results with respect to the abstract mathematical concept of *suspension*, which says that every C^1 invertible map is a Poincaré map, we introduce the concept of a *constructable* Poincaré map. Not every map is a constructable Poincaré map and this is an important distinction between dynamical synthesis and abstract nonlinear dynamics.

We also show how to use any one-dimensional map to induce a two-dimensional Poincaré map which is a completely reducible map and hence for a very broad class of maps that includes the logistic map we derive *closed form* ODEs for which these one-dimensional maps are "embedded" in a Poincaré map. This provides an avenue for the study of one-dimensional maps, such as the logistic map, as two-dimensional Poincaré maps that arise from square-wave forced electronic circuits.

1. Introduction

This paper provides a mathematical basis for the construction of a wide array of closed form Poincaré maps and in the process provides some answers to the question of when a map is a Poincaré map for an ODE that can be written down in closed form. For example, we show how to derive two-dimensional square-wave forced ODEs whose Poincaré maps are: the Chirikov or standard map, see Arrowsmith & Place [1990]; the gingerbread map of Robert Devaney [1984]; and a special case of the Hénon map [Hénon, 1969]. We show how to derive a three-dimensional square-wave forced ODE whose Poincaré map is the Hénon map. Further, our constructions show the relationship between these maps and the twist-and-flip map. Of interest is that all of the ODEs we derive can be implemented in electronic circuits.

It is well known that every map is the time-1 map of some flow on some manifold. The technique for deriving this flow is known as a *suspension* [Nitecki, 1971]. Having this abstract mathematical technique available however is of no use in actually constructing the ODE for the flow. Further the "surface" on which the flow exists cannot be easily understood except in special cases such as the Klein bottle and the Möbius strip; see Hocking & Young [1961]. In any case, the technique of suspension cannot, in general, be used to derive differential equations for electronic circuits or other applied dynamical systems as can be done with dynamical synthesis, or "constructable nonlinear dynamics."

In addition to deriving closed form ODEs for electronic circuits for many familiar maps, we show how to relate one-dimensional maps to two-dimensional invertible maps which are Poincaré maps for closed form ODEs. This connection suggests that it is possible to combine the theory of one-dimensional maps with the theory of differentiable dynamics, and thus to have a single definition of chaos that covers both invertible and noninvertible dynamical systems. As with the Chirikov and other maps, the two-dimensional maps induced by the one-dimensional maps are Poincaré maps for differential equations for electronic circuits.

Coincidentally, our constructions allow us to derive an electronic circuit for the Mandelbrot set and any other fractal set derived from noninvertible functions of a complex variable. In Chua *et al.* [1992] an example is given to show that a very broad class of *new* fractals are the limiting sets of the closure of intersecting stable and unstable manifolds. From these results there begins to emerge the outline of a theory which suggests that fractals are the discontinuous analogs of intersecting stable and unstable manifolds. Put more simply, a fractal may be the discontinuous analog of an unstable manifold having horseshoes. With these examples we suggest that the chaos in one-dimensional interval maps, fractals, and invertible dynamical systems having horseshoes can possibly be brought under the umbrella of a single, perhaps unified "theory of chaos."

Adding to this hypothesis are the results in Brown [1992] demonstrating a direct linkage between one-dimensional interval maps and three-dimensional dynamical systems such as the Chua circuit, the Lorenz equation and the Rössler equation. The results in this paper carry this linkage further to include a relation between two-dimensional square-wave forced equations and the Chua equation through the medium of one and two-dimensional maps.

The key mechanism that allows us to construct Poincaré maps is called "factorization" theory. In this theory we seek to factor Poincaré maps (see Brown & Chua [1991]) into simple parts each of which can be associated with a simple, autonomous ODE. This is presented in Sec. 2. In Sec. 3 we identify four maps that are in some intuitive sense the best or the simplest maps to occur as building blocks of Poincaré maps in $\mathbf{R}^2$. In Sec. 4 we present our theory of constructive Poincaré maps and a rigorous concept of irreducible nonlinear maps which cannot be made simpler without removing the nonlinearity. The composition of a finite number of irreducible maps is called a completely reducible map and a map which can be approximated pointwise by completely reducible maps is called a reducible map. Key features of irreducible maps are:

(1) They cannot produce chaos or any complex nonlinear effect.
(2) They can be implemented in an electronic circuit.
(3) There are only a finite number of families of irreducible maps in any n-dimensional space.
(4) Most familiar maps that produce chaos are either completely reducible or reducible.

2. Factorization Theory

In this section we give some answers to the question of when maps can be factored into a composition of simpler maps, which are Poincaré maps for constructable ODEs. A basic construction which is a generalization of the construction of Lemma 25 of Brown & Chua [1991] that will recur frequently is the following: Consider the following n-dimensional ODE:

$$\dot{X} = s_1(t)F_1(X) + s_2(t)F_2(X) + s_3(t)F_3(X)\,, \quad (1)$$

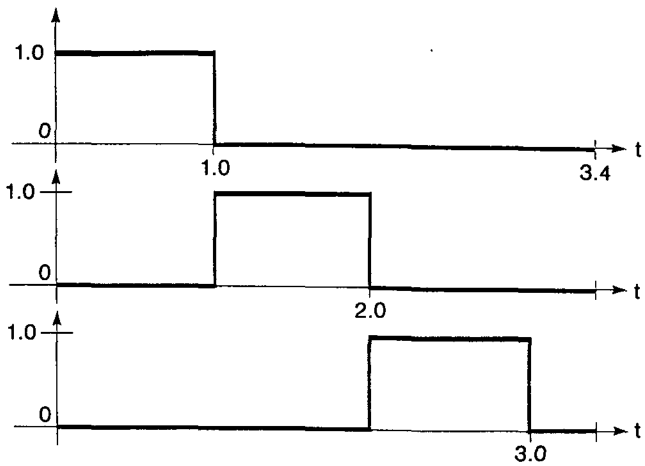

Fig. 1. The functions $s_i(t)$, $i = 1, 2, 3$, constitute a three-phase gate function. Each is periodic of the same period. $s_2(t)$ and $s_3(t)$ are phase shifts of the function $s_1(t)$.

where the $F_i(X)$ are any continuous functions in $\mathbf{R}^n$, and the functions $s_i(t)$ constitute a three-phase gate function and thus take only the values 1 or 0, are periodic, have only two discontinuities, and at any given time, only one is nonzero; see Fig. 1. Thus s_1 is 1 for $0 \le t < \tau$, and is 0 for $\tau \le t < 3\tau$ and $s_1(t + 3\tau) = s_1(t)$. $s_2(t) = s_1(t - \tau)$, and $s_3(t) = s_1(t - 2\tau)$. This gate function can easily be generated by an electronic "clock" circuit.

Thus, at time t, this ODE looks like

$$\dot{X} = F_i(X)\,, \qquad (2)$$

where we assume at time t the function $s_i(t) = 1$. The Poincaré map for Eq. (1) is the composition of three maps determined by the three solutions of Eq. (2), for $i = 1, 2, 3$. In particular, when $F_2 = F_3 = 0$, the equation

$$\dot{X} = s_1(t)F_1(X)$$

is a square-wave forced ODE whose Poincaré map is just the solution of this equation at time τ. In this way we are able to construct Poincaré maps from autonomous ODEs whose solutions we know in advance. It is also possible to carry our entire construction over to nonautonomous ODEs. To do this we must use the sawtooth function

$$p(t) = t \operatorname{mod}(\tau)$$

in conjunction with the square-pulse function to obtain the analog of Eq. (1):

$$\begin{aligned}\dot{X} = s_1(t)F_1(X, p(t)) + s_2(t)F_2(X, p(t))\\ + s_3(t)F_3(X, p(t))\,, \qquad (3)\end{aligned}$$

The sawtooth function can be easily generated by a standard electronic circuit found in all television sets. In order to simplify our exposition we will not consider the nonautonomous case suggested by Eq. (3) in this paper.

In this paper we also seek to answer the question: given a map, when is it a Poincaré map or a first return map for some periodically forced constructable ODE? Given the construction of Eq. (1), this reduces to the question of when a map arises from the composition of maps which in turn arise from the solution of an autonomous constructable ODE.

2.1. *Factorization lemmas*

We first prove a lemma that tells us when a family of maps solves an ODE.

Lemma 1. *Let $g(\mathbf{x}_0, t)$ be a one-parameter family of maps, $\mathbf{x}_0 \to g(\mathbf{x}_0, \cdot)$ from $\mathbf{R}^n$ to $\mathbf{R}^n$ with the following properties:*

(1) *For each $\mathbf{x}_0 \in \mathbf{R}^n$, the function $\mathbf{x}(t) = g(\mathbf{x}_0, t)$ is differentiable with respect to t.*
  (1) *$g(\mathbf{x}_0, 0) = \mathbf{x}_0$, for all $\mathbf{x}_0$.*
  (2)
$$\dot{\mathbf{x}} = \frac{\partial g(\mathbf{x}_0, t)}{\partial t}$$
  exists for all $(\mathbf{x}_0, t)$.

(2) *$g^{-1}(\mathbf{x}_0, t)$ exists for every t.*
  Then for fixed $\mathbf{x}_0$, the function $\mathbf{x}(t) = g(\mathbf{x}_0, t)$ solves the initial value problem:

$$\dot{\mathbf{x}} = \frac{\partial g(g^{-1}(\mathbf{x}, t), t)}{\partial t}\,, \qquad \mathbf{x}(0) = \mathbf{x}_0\,. \qquad (4)$$

Proof. Since $g^{-1}(\mathbf{x}, t) = \mathbf{x}_0$ for all t we have

$$\dot{\mathbf{x}} = \frac{\partial g(\mathbf{x}_0, t)}{\partial t} = \frac{\partial g(g^{-1}(\mathbf{x}, t), t)}{\partial t}\,. \quad \blacksquare$$

Example. Let $n = 1$ and $x(t) = g(x_0, t) = x_0 \exp(t)$. Then for fixed t, $g^{-1}(u, t) = u \exp(-t)$. Hence, $\dot{x} = x_0 \exp(t) = g^{-1}(x, t) \exp(t) = x \exp(-t) \exp(t) = x$.

Example. Let

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0 \exp(t) \\ y_0 + f(x_0 \exp(t)) \end{pmatrix}.$$

Then $\dot{x} = x$ and $\dot{y} = f'(x)\dot{x}$. The invertibility requirement means that the initial conditions, (x_0, y_0) can be eliminated from the equation for the derivative and thus we can obtain an equation between the function and its derivative having no parameters or arbitrary constants.

Lemma 2. *Given a one-parameter family of maps, $g(\mathbf{x}_0, t)$, satisfying the conditions of Lemma 1 above and for which*

$$\frac{\partial g}{\partial t}(g^{-1}(\mathbf{x}, t), t)$$

is independent of t, then for $t = \tau$, the map $\Phi_\tau(\mathbf{x}) = g(\mathbf{x}, \tau)$ is a Poincaré map for some constructable square-wave forced ODE.

Proof. The equation is given by

$$\dot{x} = sg(\omega t)\left(\frac{\partial g}{\partial t}(g^{-1}(\mathbf{x}, t), t)\right),$$

where $sg(\omega t) = 0.5(1 + \mathrm{sgn}(\sin(\omega t)))$ and $\omega = \pi/\tau$. ■

Let us extend this idea to a finite number of maps satisfying the conditions of Lemma 2.

Lemma 3. *Given maps $\mathbf{T}_1, \mathbf{T}_2, \ldots, \mathbf{T}_n$ each satisfying the conditions of Lemma 2 there exists a square-wave forced ODE for which their composition is the Poincaré map.*

Proof. The proof consists in using the preceding lemma in extending the construction of Eq. (1) extended to n terms. ■

Definition (Time-1 Map). The family of maps $g(\mathbf{x}_0, t)$ represents the general solution of the differential equation, Eq. (4). For $t = 1$, the map $\Phi(\mathbf{x}_0) = g(\mathbf{x}_0, 1)$ is called the time-1 map.

The question of when a specific map is a Poincaré map of an ODE which can be written down in closed form can now be reduced to the following question: *When can a map be factored into components, each of which is a time-1 map for some constructable ODE?*

We now apply these ideas to construct a square-wave forced ODE whose Poincaré map is the Hénon map. We do this in two steps, first for a special case, then for the general case.

Lemma 4 (The Hénon Map). *Consider the map,*

$$x \to 1 - ax^2 + y,$$
$$y \to bx,$$

where a, b are constants. If $b < 0$ then this map is the time-1 map of a closed form ODE.

Proof. The preceding lemmas say that if we are able to factor this map into a composition of maps, each of which is a time-1 map of an autonomous ODE, then there is a square-wave forced ODE for which the Hénon map is the Poincaré map.

We construct this factorization into time-1 maps explicitly along with the one-parameter family of maps, $g(\mathbf{x}_0, t)$, and the ODEs:

1. The factors (time-1 maps):

$$T_1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -x \end{pmatrix},$$

$$T_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 1 - ay^2 \\ y \end{pmatrix},$$

$$T_3\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ by \end{pmatrix}.$$

2. The one-parameter family of maps, g:

$$T_1\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos\left(\frac{\pi t}{2}\right) & \sin\left(\frac{\pi t}{2}\right) \\ -\sin\left(\frac{\pi t}{2}\right) & \cos\left(\frac{\pi t}{2}\right) \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix},$$

$$T_2\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0 + (1 - ay_0^2)t \\ y_0 \end{pmatrix},$$

$$T_3\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0 \\ \exp(\alpha t)y_0 \end{pmatrix}.$$

3. The equations:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \frac{\pi}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix},$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 - ay^2 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha y \end{pmatrix}.$$

By comparing the composition of the maps T_i with the Hénon map, we find that in this case $b = -\exp(\alpha)$, and hence $b < 0$.

The final ODE for the Hénon map with $b < 0$ is given by

$$\dot{X} = s_1(t)F_1 + s_2(t)F_2 + s_3(t)F_3 \,,$$

where the vector fields F_i are obtained from the above ODEs, and the functions $s_i(t)$ constitute a three-phase gate function. ■

The case where $0 < b$ is more difficult:

Lemma 5. *The general Hénon map cannot be realized as the time-1 map for a closed form ODE on* $\mathbf{R}^2$.

Proof. The determinant of the Jacobian of the Hénon map is $-b$. For $b > 0$ this determinant is negative, and hence the map is orientation-reversing. By Proposition 1.6.3 of Wiggins [1988] the Hénon map cannot be a Poincaré map on $\mathbf{R}^3$. ■

A less sophisticated argument for this proof consists in observing that the determinant of the derivative matrix is a continuous function of its entries, and since it is positive at $t = 0$, and always invertible, it can never be negative since it would have to be zero at some point for this to occur. Hence every time-1 map must have a positive determinant of its derivative matrix.

As a consequence of this lemma the Hénon map for $b > 0$ is not a two-dimensional Poincaré map for a constructable ODE. Using the technique of suspension, however, it can be realized as a time-1 map of a flow on a highly complex, "nonorientable" two-dimensional surface in $\mathbf{R}^3$.

Regardless of these complications we do have the following lemma:

Lemma 6. *For $b > 0$ the Hénon map is a Poincaré map for a constructable ODE in* $\mathbf{R}^3$.

Proof. The key to this construction is the factor

$$T_1(x, y, z) = (y, x, -z) \,,$$

which is the time-1 map of the general solution of a three-dimensional autonomous ODE. Hence the time-1 maps, general solutions, g and ODE need only be specified for the first map, T_1, and we lift the other two maps to three dimensions by letting the third coordinate be constant. Doing this we have the following:

1. The factors (time-1 maps):

$$T_1\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ x \\ -z \end{pmatrix},$$

$$T_2\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+1-ay^2 \\ y \\ z \end{pmatrix},$$

$$T_3\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ by \\ z \end{pmatrix}.$$

2. The one-parameter family of maps, g:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = 0.5 \begin{pmatrix} 1+\cos(\pi t) & 1-\cos(\pi t) & \sqrt{2}\sin(\pi t) \\ 1-\cos(\pi t) & 1+\cos(\pi t) & -\sqrt{2}\sin(\pi t) \\ -\sqrt{2}\sin(\pi t) & \sqrt{2}\sin(\pi t) & 2\cos(\pi t) \end{pmatrix} \times \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix},$$

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} x_0 + (1-ay_0^2)t \\ y_0 \\ z_0 \end{pmatrix},$$

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} x_0 \\ \exp(\alpha t)y_0 \\ z_0 \end{pmatrix}.$$

3. The ODEs:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \frac{\pi}{4}\begin{pmatrix} 0 & 0 & 2\sqrt{2} \\ 0 & 0 & -2\sqrt{2} \\ -2\sqrt{2} & 2\sqrt{2} & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 1-ay^2 \\ 0 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha y \\ 0 \end{pmatrix}.$$

The square-wave forced ODE can now be written down, parallel to the special case for negative b, as the sum of these three vector fields multiplied by the three-phase gate function. ■

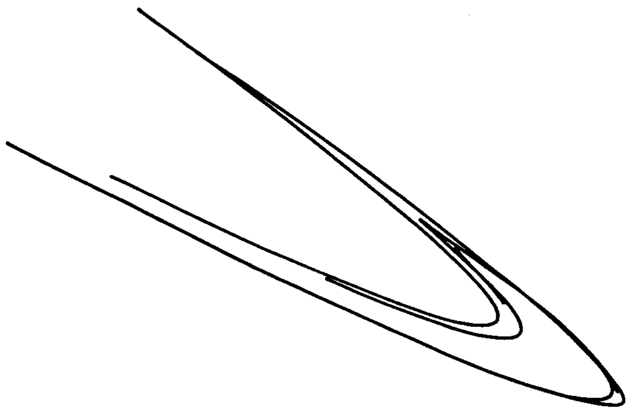

Fig. 2. The attractor that arises from the Poincaré map induced by the Hénon map. The attractor is shown in three-dimensional space and is generated by using the values $z = 0.0$, $a = 1.4$, $b = 0.3$.

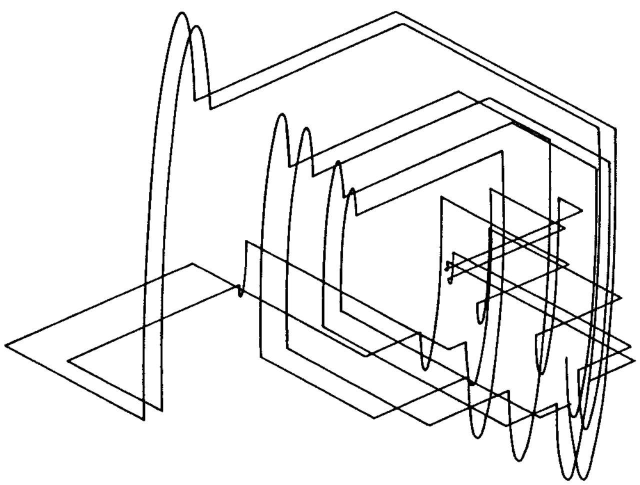

Fig. 3. The three-dimensional integral curves that arise from the forced oscillator induced by the Hénon map. In this figure $z = 0.0$, $a = 1.4$, $b = 0.3$.

This construction has the interesting effect of lifting the Hénon map to the three-dimensional space. By choosing $a = 1.4$, $b = 0.3$, $t = 1$ and $z = 0.0$ we obtain the familiar Hénon attractor in three-dimensional space; see Fig. 2. Figure 3 illustrates the integral curves that arise from integrating the square-wave forced ODE. When $z \neq 0$, we get two copies of the attractor; see Fig. 4.

In all cases, we may obtain a C^∞ equation by replacing the three-phase gate function by a C^1 function. Since all square-wave functions are built up from the sgn function, it is usually sufficient to replace the sgn function by the hyperbolic tangent, $\tanh(0.5\,\beta t)$. For large β this function is nearly the sgn function.

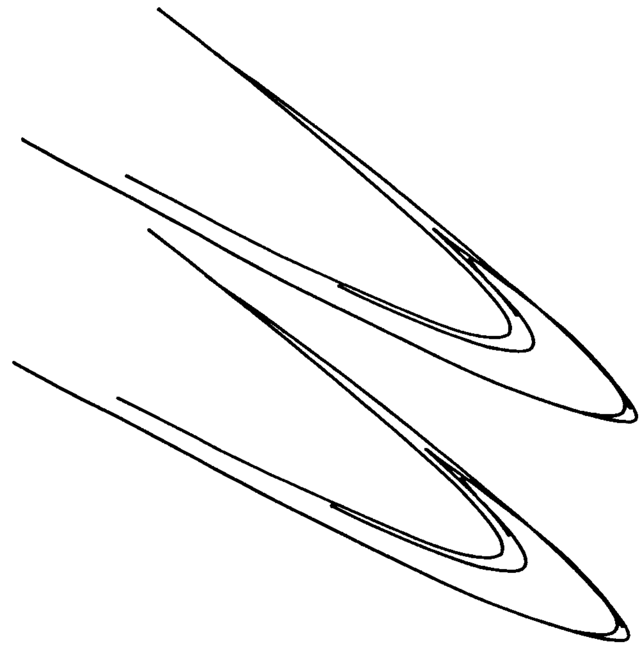

Fig. 4. The attractor that arises from the Poincaré map induced by the Hénon map occurs as a "dual" attractor when the value of z is not 0.0. Other parameter values are the same as in Fig. 3. Dual attractors also occur in maps such as Ueda's strange attractor when the Poincaré map is determined by half the frequency of the forcing function.

Lemma 7. *Given any autonomous ODE*

$$\dot{x} = G(x)\,,$$

the time-1 map of the solution of this ODE is also the Poincaré map for a square-wave forced ODE.

Proof. The square-wave forced ODE is given by

$$\dot{x} = sg(\omega t)G(x)$$

where $\pi = \omega$. ■

Lemma 8. *Given any finite number of prescribed fixed points, there exists a square-wave forced ODE whose Poincaré map has exactly those fixed points.*

Proof. Chua [1971] and the above lemma.

We repeat from Brown & Chua [1991] the factorization lemma:

Lemma 9. *Let* $\mathbf{x}(t)$ *be a solution of*

$$\dot{\mathbf{x}} = G(\mathbf{x}, t)\,, \quad \textit{and} \quad \mathbf{x}(0) = \mathbf{x}_0\,, \tag{5}$$

where $(\mathbf{x}_0, t) \in \mathbf{R}^n \times \mathbf{R}$. *Also, suppose that there exists a constant* p *such that for all* $(\mathbf{x}, t) \in \mathbf{R}^n \times \mathbf{R}$

the function G satisfies the relation

$$-G\left(\mathbf{x},\, t+\frac{p}{2}\right) = G(-\mathbf{x},\, t)\,. \tag{6}$$

Let

$$\mathbf{y}(t) = -\mathbf{x}\left(t+\frac{p}{2}\right)$$

for $0 \le t < \infty$. *Then* $\mathbf{y}(t)$ *is a solution of the equation*

$$\dot{\mathbf{y}} = G(\mathbf{y},\, t)\,, \quad \textit{and} \quad \mathbf{y}(0) = -\mathbf{x}\left(\frac{p}{2}\right).$$

Proof. Brown & Chua [1991].

The function G in Eq. (5) is periodic as a function of t with period p. If we let T_t be the one-parameter family of phase plane diffeomorphisms in $\mathbf{R}^2$ defined by Eq. (5), then the Poincaré map is T_p. We use this fact in the following corollary.

Corollary 1. *Let* $\mathbf{F}$ *be a 180-degree rotation about the origin in* $\mathbf{R}^2$ *and let* T_t *be the one-parameter family of phase plane diffeomorphisms defined by Eq.* (5). *Then the Poincaré map for Eq.* (5), *i.e.,* T_p, *is given by the diffeomorphism,* Φ *of* $\mathbf{R}^2$, *defined by*

$$\Phi(\mathbf{x}_0) = FT_{p/2}FT_{p/2}(\mathbf{x}_0)\,.$$

Proof. Brown & Chua [1991].

This corollary is also true for $\mathbf{R}^n$.

We will always assume $p = 2\pi/\omega$, so we drop the subscripts and write the Poincaré map simply as $FTFT$.

Example (Construction of a Van der Pol Type Poincaré Map). The above lemmas tell us that we need only be able to construct two factors, F and T, with prescribed characteristics and their composition will be the desired Poincaré map.

A technique that will be extended to "action-angle" coordinates is to use polar coordinates to define a mapping T which must have a limit cycle. We do this in several steps and we begin with the following ODE for a twist in polar coordinates:

$$\dot{r} = 0\,,$$
$$\dot{\theta} = r\,.$$

More generally we have for arbitrary but integrable $f(r)$

$$\dot{r} = 0\,,$$
$$\dot{\theta} = f(r)\,.$$

If damping is added we have

$$\dot{r} = \beta r\,,$$
$$\dot{\theta} = f(r)\,.$$

And to obtain a limit cycle, we use

$$\dot{r} = \beta r(b-r)\,,$$
$$\dot{\theta} = f(r)$$

for Van der Pol type equations $f(r) = 1$.

We now transform to rectangular coordinates and shift the critical point to $(a,\, 0)$ to obtain the T equation. Replace a by $a\,\mathrm{sgn}(\sin(\omega t))$ to obtain a square-wave forced equation that has many properties of the Van der Pol equation.

To make our example specific let $\beta = 1$; then

$$\dot{r} = r(b-r)\,,$$
$$\dot{\theta} = 1\,.$$

The solution of this system is given by the relations

$$r(t) = \frac{br_0}{r_0 + (b-r_0)\exp(-bt)}\,,$$
$$\theta = \theta_0 + t\,.$$

In rectangular coordinates this solution is given by

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = f(t,\, r_0)\begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix},$$

where $f(t,\, r_0) = b/(r_0 + (b-r_0)\exp(-bt))$, and $r_0 = \sqrt{x^2+y^2}$.

The T map can be taken as

$$T_\tau\begin{pmatrix} x \\ y \end{pmatrix} = f(\tau,\, r)\begin{bmatrix} \cos(\tau) & -\sin(\tau) \\ \sin(\tau) & \cos(\tau) \end{bmatrix}\begin{pmatrix} x-a \\ y \end{pmatrix} + \begin{pmatrix} a \\ 0 \end{pmatrix},$$

where $f(\tau,\, r) = b/(r + (b-r)\exp(-b\tau))$, and $r = \sqrt{(x-a)^2+y^2}$. According to the factorization lemma, a is the amplitude of a square-wave force and τ is half the period of the forcing function, usually written as π/ω for convenience.

The map $FTFT$ is the Poincaré map for the following two-dimensional system of differential equations in rectangular coordinates:

$$\dot{x} = (b - r)(x - as(\omega t)) - y,$$
$$\dot{y} = (b - r)y + (x - as(\omega t)),$$

where $r = \sqrt{(x - as(\omega t))^2 + y^2}$, and $s(\omega t) = \mathrm{sgn}(\sin(\omega t))$.

2.2. *Topological conjugacy and groups*

In this subsection we add some results that are valuable in dynamically synthesizing Poincaré maps having given properties. One of the most important properties that need to be preserved in dynamical synthesis is that of the topological conjugacy between a Poincaré maps and its inverse. That is, if we know that the Poincaré map of a given differential equation (whose Poincaré map cannot be obtained in closed form) is topologically conjugate to its inverse, then the synthesized Poincaré map must also have this property. The following lemma tells us when this can occur in great generality.

Lemma 10 (Topological Conjugacy). *Let G be a group and let a, b, c be three elements in the group having the following relations:*

$$ca = a^{-1}c,$$
$$cb = b^{-1}c;$$

then there exists an element $g \in G$ such that

$$g(ab) = (ab)^{-1}g.$$

Proof. This follows from a direct calculation which shows that $g = ca^{-1}$ is the required element of G. ∎

This lemma applied to maps says that if two maps a and b are topologically conjugate to their inverses by the same conjugacy, c, then the composite of the two maps is also topologically conjugate to its inverse, but not generally by the same conjugacy. We have the following lemma which allows us to determine when a composition of maps is topologically conjugate to its inverse:

Lemma 11. *Let*

$$\Phi = T_{a_1} \circ T_{a_2} \circ T_{a_3} \circ \cdots \circ T_{a_n} \circ T_{a_{n-1}} \circ \cdots T_{a_2} \circ T_{a_1},$$

where $\mathbf{P} \circ T_{a_i} = T_{a_i}^{-1} \circ \mathbf{P}$ for all i; then

$$\mathbf{P} \circ \Phi = \Phi^{-1} \circ \mathbf{P}.$$

Proof. The proof is a direct computation.

Example. Let three maps of the plane be P, F, T; P is reflection about the horizontal axis, F is the flip, and T is any twist centered anywhere on the horizontal axis. Then the map FT is topologically conjugate to its inverse, with conjugacy equal to $PF = R$, where R is a reflection about the vertical axis. As a second example, consider the Hénon map with $b = -1$. The map can be written as $L\Phi$, where Φ is a 90-degree rotation, and L is a nonlinear shift. By a direct computation we have $RL = L^{-1}R$, and $R\Phi = \Phi^{-1}R$, where R is as in the preceding example, i.e. reflection about the vertical axis. Hence, the Hénon map with $b = -1$ is topologically conjugate to its inverse with conjugacy RL^{-1}. Given two twist maps T_a, T_b with critical points on the horizontal axis at $(a, 0)$ and $(b, 0)$ respectively, we know that their composition is topologically conjugate to its inverse by the map PT_a^{-1}.

Definition (Constructive Poincaré Maps). A constructive Poincaré map is a first return map or a time-1 map for a flow on an *orientable* manifold which can be expressed as composition of a finite number of factors each of which, in turn, can be expressed in terms of elementary functions or special functions that can be implemented by an algorithm on a computer or in an electronic circuit.

This definition eliminates the Möbius band, the Klein bottle and any other nonorientible surface as a surface on which a flow may be expressed.

The Constructive Poincaré Map Group. The preceding lemmas, examples and discussion suggest the following three corollaries:

Corollary 2. *If the maps $\mathbf{T}_1, \mathbf{T}_2, \ldots, \mathbf{T}_n$ are each constructive Poincaré maps, then their composition is a constructive Poincaré map, and hence the set of constructive Poincaré maps is closed under composition.*

Corollary 3. *If the map $\mathbf{T}$ is a constructive Poincaré map, then the inverse map, $\mathbf{T}^{-1}$, is a constructive Poincaré map.*

Corollary 4. *The map* **I** *is a constructive Poincaré map.*

Since the operation of composition is associative, the set of constructive Poincaré maps defined on a given domain, D, form a group, which we will call the constructive Poincaré map group, $P(D)$.

In all further discussions in this paper whenever we use the term *Poincaré map* we will mean a *constructive Poincaré map* and when we use the term *map*, we will mean a *constructive map.*

3. Four Fundamental Constructive Maps: The Twist, Shift, and Two Dilation/Contraction Maps

Our theory of constructive maps will benefit from the existence of a minimal set of basic constructive maps that form the building blocks for most constructive Poincaré maps. We present four of these maps in this section.

In analogy with Hamiltonian systems having elliptic points [Arnold & Avez, 1989] where the behavior of a measure-preserving map in the vicinity of an elliptic periodic point can often be expressed as a twist, we define the following three "fundamental" maps.

The Simple Twist

The simple twist is defined by the equation

$$T\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} r \\ \theta + r \end{pmatrix}.$$

We have written this equation suggesting polar coordinates, or better, action-angle coordinates [Wiggins, 1990], but we will use this concept in any coordinates that are convenient. In action-angle coordinates the simple twist appears as a linear function, i.e.

$$T\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} r \\ \theta \end{pmatrix}.$$

The more general twist used in Brown & Chua [1991, 1992] is given by

$$T_f\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} r \\ \theta + f(r) \end{pmatrix},$$

and may be expressed symbolically as

$$T\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ f & 1 \end{pmatrix}\begin{pmatrix} r \\ \theta \end{pmatrix},$$

where it is understood that the symbolic multiplication $f \cdot r$ is to be interpertated as $f(r)$.

The Simple Shift

The polar coordinate form of the simple shift is given by

$$T\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} r + \theta \\ \theta \end{pmatrix}.$$

This is also linear in the action-angle formulation. Note that the symbolic matrix representation of that of the simple shift is the transpose of the simple twist, and the general shift, using our symbolic notation, can thus be defined as the transpose of the general twist. We have the following lemma justifying the twist and shift maps as being the simplest possible nonlinear constructive Poincaré:

Lemma 12 (Twist and Shift Lemma). *Let*

$$T_t\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ f(x, y, t) \end{pmatrix}$$

be a one-parameter family of C^∞ invertible maps of $\mathbf{R}^2$ such that the following is true:

(1) $T_{s+t} = T_s \circ T_t$, $T_0 = \mathbf{I}$, *the identity map, and* $T_t^{-1} = T_{-t}$;
(2) $\det(D(T_t)) = 1$ *for all* t.

Then

$$T_t\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y + \Omega(x)t \end{pmatrix}$$

and T_t is a one-parameter group of twist maps.

Proof. Since

$$D(T_t) = \begin{pmatrix} 1 & 0 \\ f_x & f_y \end{pmatrix} = 1$$

we have $f_y = 1$ so that $f_y(x, y, t) = y + g(x, t)$. By the group property we have $f(x, f(x, y, t), s) = y + g(x, t) + g(x, s)$ so that $g(x, s) + g(x, t) = g(x, s + t)$. Since by our hypothesis g is continuous, and $g(x, 0) = 0$ from the group property, we have $g(x, t) = \Omega(x)t$. ■

The Dilation/Contraction Maps

The first dilation/contraction map is given by

$$T\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} r \\ \theta \exp(f(r)t) \end{pmatrix},$$

or in matrix form we have

$$T\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \exp(f(r)t) \end{pmatrix}\begin{pmatrix} r \\ \theta \end{pmatrix}.$$

The second dilation/contraction map is given by

$$T\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} r\exp(f(\theta)t) \\ \theta \end{pmatrix},$$

or in matrix form,

$$T\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} \exp(f(\theta)t) & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} r \\ \theta \end{pmatrix}.$$

We call these maps "fundamental," for the following reasons. First, for Hamiltonian systems having elliptic fixed points, the twist map characterizes the type of nonlinear dynamics that takes place near this point. Second, each of these four maps solves an autonomous ODE and they are thus building blocks for constructive Poincaré maps. Third, composition of these maps gives rise to a diverse array of dynamics that includes most forms of chaos we know. Fourth, each of these maps can be implemented in a simple electronic circuit. Fifth, each of these maps has analogs in n-dimensions. The sixth reason is that these four maps may be used to approximate the solution to a large class of ODEs.

Justifying the label of the dilation/contraction map as the "simplest" possible dilation/contraction solving an autonomous ODE, we have the following proposition:

Proposition 1 (Dilation/Contraction). *Let*

$$T_t\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ f(x, y, t) \end{pmatrix}$$

be a one-parameter family of C^∞ invertible maps of $\mathbf{R}^2$ such that the following is true:

(1) $T_{s+t} = T_s \circ T_t$, $T_0 = \mathbf{I}$, *the identity map, and* $T_t^{-1} = T_{-t}$;
(2) $\det(D(T_t)) = \exp(g(x, y, t)) > 0$ *for all* x, y, t.

Then

$$T_t\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y\exp(\Omega(x)t) + c(x)(\exp(\Omega(x)t - 1)/\Omega(x) \end{pmatrix},$$

where $c(x)$ and $\Omega(x)$ are C^∞ functions of x and T_t is a one-parameter group of dilation/contraction maps. If we are to have this family as the simplest possible, then $c(x) = 0$.

Proof. The proof of this proposition proceeds in a series of lemmas. In order to maintain continuity in the presentation we defer the proof of the lemmas to the appendix, but we present the statement of the four lemmas here:

First we prove

Lemma 13

$$f(x, y, t) = \int_0^y \exp(tg(x, \eta, t))d\eta + a\int_0^t \exp(\eta g(x, 0, \eta))d\eta.$$

Next we demonstrate

Lemma 14. *For all a, s, x, y we have*

$$\begin{aligned} a\exp(sg(x, y, s)) &= a\exp(s, g(x, 0, s)) \\ &\quad + \int_0^y \exp(sg(x, \eta, s)) \\ &\quad \times \left(g(x, \eta, s) + s\frac{\partial g(x, \eta, s)}{\partial t}\right)d\eta. \end{aligned}$$

Solving a PDE for g we get

Lemma 15

$$g(x, y, t) = F(x, at + y).$$

Finally we show that F is a function only of x:

Lemma 16

$$F(x, u) = \Omega(x). \quad ■$$

Remark. As $\Omega(x) \to 0$, the conclusion of the dilation/contraction lemma then reduces to the conclusion of the twist and shift lemma. Further, the map T can be written as a composition of a simple dilation/contraction map and a twist map. Hence, when $c(x) = 0$, the map T can be reduced to simpler components, i.e. $T = T_2 \circ T_1$, where

$$T_1 = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y\exp(\Omega(x)t) \end{pmatrix}$$

and

$$T_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y + c(x)(\exp(\Omega(x)t) - 1)/\Omega(x) \end{pmatrix}.$$

The map T_2 however is not precisely what we are calling a twist map.

We now provide examples to illustrate how these four maps can be used to construct a wide array of known dynamics. We first note that our demonstration of how to make the Hénon map a Poincaré map was based on factoring the Hénon map into a composition of these four maps. We now provide four more examples.

3.1. *The Chirikov (standard) map*

Since we know that a twist-and-shift can produce chaos (see Brown & Chua [1991]), we compose these two maps to obtain what is possibly the simplest chaotic map that can be written down:

$$T\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} 2r+\theta \\ r+\theta \end{pmatrix}.$$

In matrix form this map is

$$T\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} r \\ \theta \end{pmatrix}.$$

Considering this otherwise unbounded map in mod(1) addition we have the familiar "cat map." It can be made a constructive Poincaré map; however, the mod(1) addition [essentially, using addition mod(1) means that a map requires three dimensions to be defined] is a complication we do not need to retain given that we can obtain useful two-dimensional analogs of the "cat map" by choosing the factors in our composition more carefully. If we choose a shift that has both positive and negative values then the resulting twist-and-shift becomes dynamically more like that of the twist-and-shift in Brown & Chua [1991] and the modulo addition can be eliminated. Doing this in an obvious way leads us to the map of Chirikov.

The most general composition of a twist map and a shift map in polar coordinates is given by

$$T\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} r+f(\theta+g(r)) \\ \theta+g(r) \end{pmatrix},$$

and we know from the above lemmas that any such map is a constructive Poincaré map. In rectangular coordinates this map is given by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1+f(\theta+g(r))}{r} \times \begin{pmatrix} \cos(g(r)) & -\sin(g(r)) \\ \sin(g(r)) & \cos(g(r)) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

The two components of the square-wave forced ODE for which this map is the Poincaré map are given by

The shift ODE:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \frac{f(\theta)}{r}\begin{pmatrix} x \\ y \end{pmatrix},$$

where $\theta = \arctan(y/x)$ and $r = \sqrt{x^2+y^2}$, and

The twist ODE:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = g(r)\begin{pmatrix} -y \\ x \end{pmatrix}.$$

The square-wave forced ODE is given by

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = sg(\omega t)g(r)\begin{pmatrix} -y \\ x \end{pmatrix} + \frac{(1-sg(\omega t))f(\theta)}{r}\begin{pmatrix} x \\ y \end{pmatrix},$$

where, as above, $sg(\omega t) = \frac{1}{2}(1+\operatorname{sgn}(\sin(\omega t)))$. The *Chirikov map, or standard map*, is seen to be a special case of the twist-and-shift; thus we have the following lemma:

Lemma 17. *The Chirikov map is a constructive Poincaré map.*

Proof. The Chirikov map is a composition of two constructive time-1 maps, a shift map with $g(r) = r$, and a twist map with $f(\theta) = -K\sin(\theta)$. The composition of these two maps is given by

$$T\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} r-K\sin(\theta) \\ \theta+r-K\sin(\theta) \end{pmatrix},$$

which is the Chirikov map. ∎

We easily concluded that the Chirikov map is a constructive Poincaré map: however, it is usually considered with mod(1) arithmetic, although this is not necessary. By interchanging the order of the factors and making this map a twist-and-shift instead of a shift-and-twist, we can construct another useful map that is equivalent in complexity to the Chirikov map, but has some computational advantages. In rectangular coordinates the Chirikov twist-and-shift map, as we shall call this map, is

$$T\begin{pmatrix} x \\ y \end{pmatrix} = (1+K(\sin(r+\theta)/r)) \times \begin{pmatrix} \cos(r) & -\sin(r) \\ \sin(r) & \cos(r) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix},$$

where $\sin(r+\theta) = (y\cos(r) + x\sin(r))/r$. If we label the shift L and the twist T this map is written as LT, whereas the usual Chirikov map would be written as TL^{-1}. These two maps are topologically conjugate. If P is a reflection about the horizontal axis we have $PL = L^{-1}P$ by a direct computation. From Brown & Chua [1991], $PT = T^{-1}P$. Hence, by the conjugacy lemma (Lemma 6 above), LT is topologically conjugate to its inverse with the conjugacy map equal to PL^{-1}. This map parallels in all key aspects the properties of the twist-and-flip and we should expect a version of the horseshoe twist theorem [Brown & Chua, 1991] to be true for this map as well.

We now turn to another map that has appeared in the literature, the gingerbread map of Devaney [1984].

Lemma 18 (The Gingerbread Map). *The gingerbread map, given by the equation*

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 1 - y + |x| \\ x \end{pmatrix},$$

is a constructive Poincaré map.

Proof. The gingerbread map is the composition of two maps that are time-1 maps for an autonomous ODE. A 90-degree rotation,

$$T_1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix},$$

and the shift map,

$$T_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Hence it is a constructive Poincaré map. ■

Example. As an example we write down a square-wave forced ODE for which this map is the Poincaré map:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = sg(t)\begin{pmatrix} 0 & f \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + 0.5\pi(1 - sg(t))\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix},$$

where $sg(t) = (\mathrm{sgn}(\sin(\omega t)) + 1)/2$, with $\omega = \pi$ and $f(u) = 1 + |u| = 1 + \mathrm{sgn}(u)u$. If we replace $\mathrm{sgn}(u)$ with $\tanh(0.5\beta u)$, then this equation can be made C^∞. We call this the gingerbread ODE. By including a factor multiplying the function $\mathrm{sgn}(\sin(\omega t))$ this equation can be made the basis of a signal detection device [Brown & Chua, 1992]. If we carry out the necessary calculations we obtain the following square-wave forced ODE for the signal detection device:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = 0.5\begin{pmatrix} f(y) - 0.5\pi y \\ 0.5\pi x \end{pmatrix} + 0.5a\,\mathrm{sgn}(\sin(\omega t))\begin{pmatrix} -(f(y) + 0.5\pi y) \\ 0.5\pi x \end{pmatrix},$$

where a is the amplitude of the square wave, and ω is the frequency. A signal detector is obtained by adding an input signal amplitude to a. ■

3.2. *One-dimensional maps*

In this subsection we show how to "embed" any one-dimensional C^1 map (a C^1 function from $\mathbf{R}^1$ to $\mathbf{R}^1$) in a two-dimensional constructive Poincaré map.

We need a precise definition of embedding in this case:

Definition (Embedding). A map $f : \mathbf{R} \to \mathbf{R}$ is said to be embedded in a family of maps, $T_b : \mathbf{R}^2 \to \mathbf{R}^2$, if T_0 maps all of $\mathbf{R}^2$ onto the graph of f in $\mathbf{R}^2$. In this case we say that the map f induces the family of maps T_b.

An interesting application of this technique is to the logistic map, where the presence of chaos in the logistic map can be seen to be related to the existence of horseshoes in the related two-dimensional Poincaré map.

Since we are seeking to construct a Poincaré map we need only construct the map from the composition of maps which solve autonomous ODEs, and by the preceding lemmas we are assured that it is a constructive Poincaré map. We are likewise assured that they can be implemented in an electronic circuit. Since the four fundamental maps are all solutions of autonomous ODEs, it is reasonable to try to do this with these four maps. We have the following lemma:

Lemma 19. *Given any map $f : \mathbf{R} \to \mathbf{R}$ which is C^1 it can be embedded in a constructive Poincaré map on $\mathbf{R}^2$.*

Proof. We choose our components as follows:

The factors

$$T_1\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} r \\ b\theta \end{pmatrix},$$

where $0 < b < 1$,

$$T_2\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} r \\ \theta + r \end{pmatrix},$$

$$T_3\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} r + f(\theta) - \theta \\ \theta \end{pmatrix}.$$

The composition of these three maps is given by

$$T_b\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} f(b\theta + r) - b\theta \\ b\theta + r \end{pmatrix};$$

since these three maps are all constructive time-1 maps for autonomous ODEs, we know that the map we have produced is a constructive Poincaré map. We now show that the one-dimensional map is embedded in this map:

Letting $b \to 0$ in the above map we obtain the map

$$T_0\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} f(r) \\ r \end{pmatrix},$$

which maps $\mathbf{R}^2$ onto the graph of function f. ■

Since all maps used in the above embedding are elements of a group we may invert T_b by forming the individual inverses, which is very easy. The inverse of T_b is given by $T_1^{-1} \circ T_2^{-1} \circ T_3^{-1}$, which is

$$T_b^{-1}\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} r + \theta - f(\theta) \\ (f(\theta) - r)/b \end{pmatrix}.$$

Note that the invertiblity of f is not relevant to the invertibility of T_b, and also that the inverse of T_b does not depend on the continuity of f. We now show that this particular embedding is very useful and natural. The fixed points of T are determined by those of f since if

$$\begin{pmatrix} f(b\theta + r) - b\theta \\ b\theta + r \end{pmatrix} = \begin{pmatrix} r \\ \theta \end{pmatrix};$$

then

$$f(b\theta + r) = b\theta + r \quad \text{and} \quad r = (1 - b)\theta,$$

so that we must have $f(\theta) = \theta$.

The derivative of T is given by the matrix

$$\begin{pmatrix} f'(b\theta + r) & b(1 - f'(b\theta + r)) \\ 1 & b \end{pmatrix}.$$

The determinant is b and the trace is $b + f'(b\theta + r)$. As $b \to 0$ this determinant becomes 0. At a fixed point the derivative is

$$\begin{pmatrix} f'(\theta) & b(1 - f'(\theta)) \\ 1 & b \end{pmatrix}.$$

If $b < 1$, and $|f'(\theta)| > 1$, then T is hyperbolic at this fixed point, which intuitively agrees with the behavior of f at its fixed point.

Example. Take f as the logistic map. Then $f(x) = \mu x(1 - x)$. The nonzero fixed point is given by

$$x = \frac{\mu - 1}{\mu}.$$

Since we are only interested in fixed points between 0 and 1, we must have $\mu > 1$. If $|f'(\theta)| > 1$, we must have $\mu > 3$. Thus when $\mu > 3$, T has a hyperbolic fixed point at $((1 - b)\theta, \theta)$, where $\theta = (\mu - 1)/\mu$.

Figures 5–7 are illustrations of the unstable manifold for the Poincaré map associated with the logistic map. Figure 5 shows the stable manifold (large fat arrow) and the unstable manifold (horizontal, slender arrow). In this figure, $b = 0.15$ and $\mu = 3.89$. The large dot at an intersection of the stable and unstable manifolds indicates the location of the fixed point. Figure 6 shows the unstable manifold for $b = 0.01$ and $\mu = 3.89$. Figure 7 shows the unstable manifold for $b = 0.001$ and $\mu = 3.99$.

Numerical experiments suggest that when f is in the chaotic regime of the parameter μ, the map T has a horseshoe. This suggests that the source of chaos in one-dimensional maps can be traced to horseshoes. If all of this can be proven, the definition of chaos in one-dimensional maps could be reduced to the existence of horseshoes in the related two-dimensional Poincaré map.

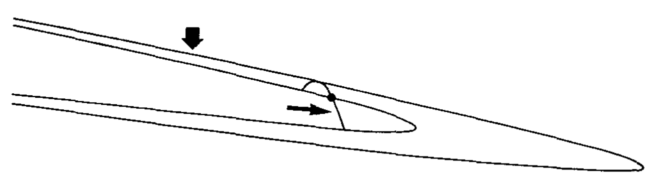

Fig. 5. The stable and unstable manifolds that arise from embedding the logistic map in a two-dimensional Poincaré map. The stable manifold is indicated by the wide arrow. The unstable manifold is indicated by the slender arrow. The large dot indicates the location of the fixed point. The logistic map parameter is 3.89, which is in the chaotic region. The embedding parameter b is 0.15. From the illustration there appear to be horseshoes in this Poincaré map.

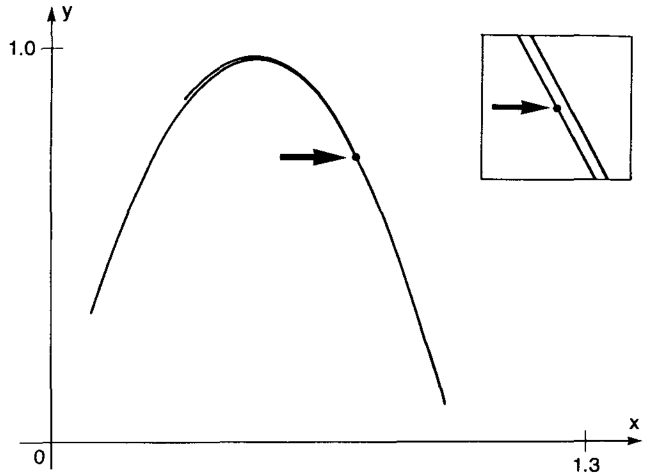

Fig. 6. The unstable manifold that arises from embedding the logistic map in a two-dimensional Poincaré map. The unstable manifold is nearly taking the shape of the logistic curve. The logistic map parameter is 3.89, which is in the chaotic region. The embedding parameter b is 0.01. The dot is the location of the fixed point. The unstable manifold seems to fold back and forth onto the graph of the logistic function.

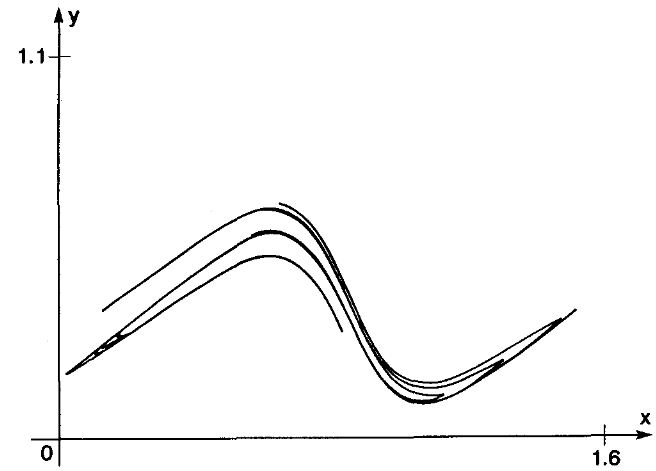

Fig. 8. The unstable manifold that arises from embedding an approximation to the shift function $2x \bmod (1)$ in a two-dimensional Poincaré map. The approximating parameter β is 15.0. The embedding parameter b is 0.15. At these parameter values the unstable manifold is only suggestive of the graph of $2x \bmod (1)$.

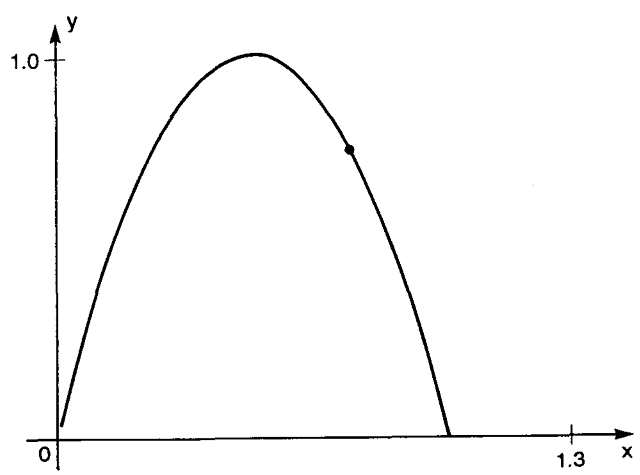

Fig. 7. The unstable manifold that arises from embedding the logistic map in a two-dimensional Poincaré map. The unstable manifold appears to have taken the complete shape of the logistic curve. The logistic map parameter is 3.99, which is in the chaotic region, and is nearly equivalent to the function $2x$ mod(1) with this parameter value. The embedding parameter b is 0.001. The dot is the location of the fixed point.

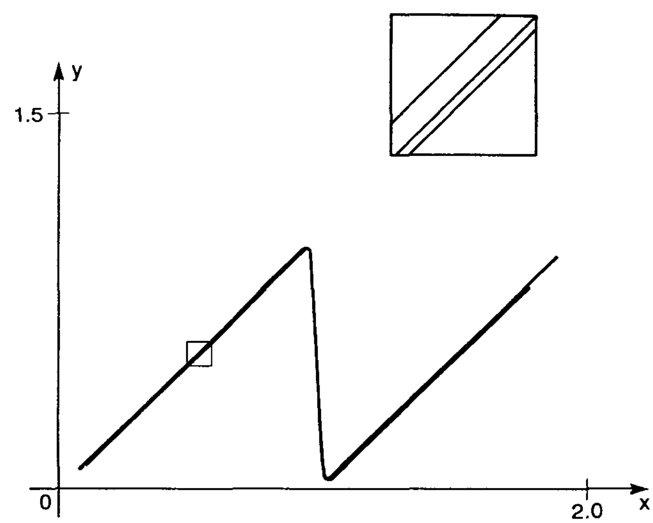

Fig. 9. The unstable manifold that arises from embedding an approximation to the shift function $2x \bmod (1)$ in a two-dimensional Poincaré map. The approximating parameter β is 150.0. The embedding parameter b is 0.01. At these parameter values the unstable manifold is near the shape of the graph of $2x \bmod (1)$ but some folds are still visible.

Example. Take f as a C^∞ analog of the function x mod(1), i.e.

$$f(x) = x - 0.5(1 - \tanh(0.5\beta(1 - x)))\,,$$

and then evaluate it at $2x$, i.e. we use the map $f(2x)$. As $\beta \to \infty$ this function converges pointwise to $2x \bmod 1$ except at $x = 1$. Figure 8 shows the unstable manifold for this map for $\beta = 15$ and $b = 0.15$. Figure 9 shows the unstable manifold for $b = 0.01$ and $\beta = 150$. Figure 10 shows the unstable manifold for $b = 0.0000000000001$ and $\beta = 150$.

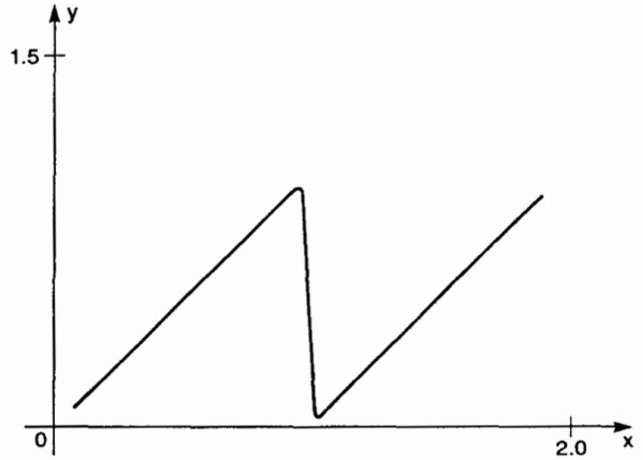

Fig. 10. The unstable manifold that arises from embedding an approximation to the shift function $2x \bmod(1)$ in a two-dimensional Poincaré map. The approximating parameter β is 150.0 The embedding parameter b is 0.000000000001. At these parameter values the unstable manifold has nearly converged to the shape of the graph of $2x \bmod(1)$.

3.3. *One-dimensional maps and the modified chua equation*

In Brown [1992] and Brown & Chua [1992b], we presented a vast array of one-dimensional maps that arise from the modified Chua equation [Brown, 1992]. Misiurewicz [1992] demonstrated that unimodial interval maps could be obtained from the "single scroll" that is obtained from the modified Chua equation. He also showed that these maps produced chaos. These results have demonstrated that the one-dimensional maps that arise from the modified Chua equations are an important class of maps to study. From these maps new insights into the nature of chaos have been obtained. In particular, they show that chaos is a continuum of effects ranging from what is known as *weak mixing* in ergodic theory (see Walters [1982]) to the horseshoe of Smale. The horseshoe in ergodic theory is known as a Bernoulli automorphism and represents the mathematical realization of a coin toss. Chaos is everything between these two categories of ergodic theory, and perhaps even more. At the upper end is the shift, or horseshoe. A simple one-dimensional example is the one-sided shift $T(x) = 2x \bmod(1)$. This is the basis of pseudo-random number generators. If we seek a generalization of the one-sided shift it would be the map $T_g(x) = g(x) \bmod(1)$. When $g(x) = 2x$ we have the one-sided shift. However, when $g(x) = a\cos(bx + c)$, where a, b, c are any real numbers with $|a| > 1$, we have a one-dimensional map related to the one-dimensional maps that arise in the modified Chua equation. Every function $g : \mathbf{R} \to \mathbf{R}$ will produce a one-dimensional map, T_g, of the unit interval. The diversity of these maps is enormous and provides a framework within which to understand chaos. We now demonstrate that these maps can be embedded in a Poincaré map.

The key to the embedding is to note that $T_g(x)$ is the composition of two functions, $g(x)$ and $x \bmod 1$. If we define, for convenience of the exposition, $m(x) \equiv x \bmod 1$ then $T_g(x) = m(g(x))$. This factorization means that we need only show how to realize $m(x)$ as a limit of differentiable functions depending on g to use the results of the previous section. To aid in this derivation we note that $m(x) = x - [x]$, where $[x]$ is the integer part of x. For $[x]$ we have the following infinite series:

$$[x] = -0.5 \sum_{k=1}^{\infty} (\mathrm{sgn}(k - x) - 1)\,.$$

Since this expression for the integer part of x is made up of all differentiable functions with the exception of the signum function, we have only to approximate the signum function by a C^∞ function to be done. But we have done this in the previous section by using the hyperbolic tangent. In order to link the convergence of the hyperbolic tangent to the convergence of the two-dimensional map associated to the one-dimensional map, we write

$$h(x) \equiv -0.5 \sum_{k=1}^{\infty} (\tanh(0.5(k - x)/b) - 1)\,,$$

so that as $b \to 0$, $h(x) \to [x]$. By the construction of the previous section as $b \to 0$ the two-dimensional map converges pointwise to the one-dimensional map. We now obtain the C^∞ function f needed to obtain the induced Poincaré map by choosing f to be

$$f(x) = g(x) - h(g(x))$$

Using the technique of the previous section, we obtain the factors of the Poincaré map induced by this one-dimensional map.

Corresponding to this family of Poincaré maps, one for each function $g(x)$, is a family of ODEs determined by using a three-phase gate in conjunction with the three factors of the Poincaré map to define a time-varying vector field which has the same

Poincaré map as a related modified Chua equation. This correspondence shows a close connection between the dynamics of the modified Chua equation and the dynamics of square-wave forced ODEs.

3.4. *Mandelbrot set circuit*

In this section we write down differential equations for a circuit for the Mandelbrot set. The equations are in two complex variables, and hence they are fourth order. We can do this with the maps defined above since the four fundamental maps can be considered as complex maps.

$$\begin{pmatrix} \dot{z} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} s_3(t)(w^2 - w + c) \\ s_1(t)\alpha w + s_2(t) z \end{pmatrix},$$

where $s_2(t) = s_1(t - 1/3)$ and $s_3(t) = s_1(t - 2/3)$ and $s_1(t)$ is periodic with period 1. On the interval $[0, 1]$, $s_1(t)$ is given by

$$s_1(t) = \begin{cases} 1 & \text{for } 0 \le t < \dfrac{1}{3}, \\ 0 & \text{otherwise}. \end{cases}$$

The parameter α is a large negative number. As $\alpha \to -\infty$, the Poincaré map for this equation converges to the map $z \to z^2 + c$. Thus for large negative α, the parameter values for which this circuit converges to a periodic solution are a point in the Mandelbrot set.

3.5. *Dynamical integration*

Consider the following ODE, which we assume has a unique solution for each pair of initial conditions:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} f_1(y)x + f_2(y) \\ g_1(x)y + g_2(x) \end{pmatrix}.$$

Given any initial conditions, the solution of this ODE, over a small enough time span, can be written as a composition of the four fundamental maps. In particular, for small time t, the composition is given by

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0 \exp(f_1(y_0)t) + f_2(y_0)t \\ y_0 \exp(g_1(x(t))t) + g_2(x(t))t \end{pmatrix}.$$

Now consider the equation

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} v - f(u) \\ -g(u) \end{pmatrix},$$

which does not appear to be in the form of the above equation. However, the vector field defined by the above equation is equivalent to one of the preceding forms by a twist determined by f, i.e. the above vector field can be twisted into one of the correct forms by the twist map

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ f & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

4. General Theory of Fundamental Maps

In this section we present several lemmas that provide a mathematical basis for the origin of the four fundamental maps discussed in previous sections, and as a result we will show that the four maps introduced in Sec. 3 are in some sense the simplest fundamental maps. In particular, these four fundamental maps are the simplest examples of a class of maps that are a natural extension of a linear vector field to a nonlinear vector field: Given any linear autonomous homogeneous vector field in $\mathbf{R}^2$, there always exists a *natural* nonlinear vector field associated to it. It is natural in the sense that it has the same integral curves and thus may have the same equilibrium points. Further, the associated nonlinear equation can be solved by linear techniques. We present lemmas that describe these fundamental nonlinear vector fields and we present an analog of the convolution theorem for the fundmental nonlinear vector fields as well. We connect the fundamental nonlinear vector fields to square-wave forced ODEs and their time-1 maps to Poincaré maps. Also we link these Poincaré maps to the conjugacy lemma of Sec. 3 through the notion of a *Lie bracket* for two matrices. Our theory can be extended to $\mathbf{R}^n$ but for simplicity of the exposition we confine our attention to the case of $\mathbf{R}^2$ here.

4.1. *Fundamental nonlinear vector fields*

The relation of the fundamental maps to the fundamental nonlinear vector fields is expressed in the following lemma:

Lemma 20 (Fundamental Maps). *Let*

$$\mathbf{z}(t) = \exp(t\mathbf{A})\mathbf{z}_0$$

be the solution of the linear autonomous homogeneous ODE

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z},$$

where

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

and

$$\mathbf{z}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} .$$

Then there exists a real-valued function $\lambda : \mathbf{R}^2 \to \mathbf{R}$ *such that* $\lambda(\mathbf{z}(t)) = \lambda(\mathbf{z}_0)$ *for all* t *along the integral curves and the function*

$$\mathbf{w}(t) = \exp(t\lambda(\mathbf{w}_0)\mathbf{A})\mathbf{w}_0$$

is the solution of the "induced" nonlinear, autonomous, homogeneous ODE

$$\dot{\mathbf{w}} = \lambda(\mathbf{w})\mathbf{A}\mathbf{w} .$$

Proof. The proof consists of directly differentiating the function $\mathbf{w}$ and noting that the integral curves of the $\mathbf{w}$-equation are the same as those of the $\mathbf{z}$-equation and hence $\lambda(\mathbf{w}(t)) = \lambda(\mathbf{w}_0)$ for all t. The function λ is obtained by integration of the equation

$$\frac{dy}{dx} = \frac{a_{21}x + a_{22}y}{a_{11}x + a_{12}y}$$

with respect to x to obtain an implicit function of x and y, $f(x, y) = C$, equal to an arbitrary constant, assuming that not both a_{11} and a_{12} are zero. We can easily integrate this equation since the substitution $y = xv$, $dy = v\,dx + x\,dv$ reduces the equation to the form $F(v)dv + dx/x = 0$, in which the variables are separable, thereby obtaining the implicit solution $g(u, v) = C$. Hence we can define the function $f(x, y) = g(y/x, x)$. The function λ is defined by $\lambda = f(x, y)$. If both a_{11} and a_{12} are zero, then $\dot{x} = 0$ and $\lambda(x, y) = x = x_0$ is a function on the integral curves $x(t) = x_0$. ■

Observe that when

$$\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} ,$$

we obtain the equation for the twist component of the twist-and-flip map. Moreover, by choosing

$$\mathbf{A} = \begin{pmatrix} 0 & -\omega^2 \\ 1 & 0 \end{pmatrix} ,$$

we obtain a twist that takes place on ellipses.

Let us extend this lemma to cover arbitrary functions of λ, $f(\lambda)$, where f is any C^1 function of $\mathbf{R}^1$ to $\mathbf{R}^1$ for which the corresponding induced ODE has unique solutions.

We first present the following technical lemma, which is adapted from Coddington & Levinson [1955]:

Lemma 21 (Technical Lemma). *Let*

$$\Phi(t) = \exp(f(t)\mathbf{A}) ,$$

where $f'(t)$ *exists for all* t. *Then*

$$\Phi'(t) = f'(t)\mathbf{A}\Phi(t) .$$

Proof. The proof is a direct computation using the definition of matrix differentiation. ■

The above lemma works because the matrices $\Phi(t)$ and $\Phi'(t)$ commute [Coddington & Levinson, 1965]. We now present our lemma on fundamental maps:

Lemma 22. (General Fundamental Maps.) *Let*

$$\mathbf{z}(t) = \exp(tf(\lambda(\mathbf{z}_0))\mathbf{A})\mathbf{z}_0 .$$

Then $\mathbf{z}$ *is the general solution of*

$$\dot{\mathbf{z}} = f(\lambda(\mathbf{z}))\mathbf{A}\mathbf{z} .$$

Proof. The proof follows by a direct computation using the preceding technical lemma. ■

We present one final lemma of this subsection:

Lemma 23. *Consider the linear homogeneous ODE,*

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} . \tag{7}$$

The induced nonlinear ODE

$$\dot{\mathbf{z}} = \lambda(\mathbf{z})\mathbf{A}\mathbf{z}$$

has the same equilibrium points as the above associated linear equation, Eq. (7).

Proof. Since the integral curves of both equations are given by

$$\lambda \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

their critical points must be the same. ■

Remark. If $\mathbf{z}(t) = \exp(f(\mathbf{z}_0)\mathbf{A}t)\mathbf{z}_0$ then $\dot{\mathbf{z}}(t) = f(\mathbf{z}_0)\mathbf{A}\mathbf{z}(t)$. In general, the constant $\mathbf{z}_0$ cannot be removed from this equation. However, if $f(\mathbf{z}_0) = f(\mathbf{z}(t))$ for all t then it is possible to remove the constant and obtain a vector field. This makes clear the value of the functions, $\lambda(\mathbf{z})$, which are invariant on the linear vector fields. For general functions f we have the following lemma:

Lemma 24. *If*

$$\dot{\mathbf{z}} = f(\mathbf{z}(t))\mathbf{A}\mathbf{z}$$

has a unique solution for each $\mathbf{z}_0$, *then*

$$\mathbf{z}(t) = \exp\left(\int_0^t f(\mathbf{z}(\eta))d\eta\mathbf{A}\right)\mathbf{z}_0 .$$

Proof. The proof is a direct computation.

Note that this is an implicit formula for $\mathbf{z}$.

The four fundamental maps of Sec. 3. The four fundamental maps discussed in Sec. 3 are obtained from the nonlinear equations for $\mathbf{A}$ corresponding to the four matrices

$$\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}$$

Note that these four matrices form a vector space basis for the four-dimensional vector space of all 2×2 matrices.

As a result of the importance of these four maps we make the following definition of irreducible maps:

Definition (Irreducible Time-1 Maps). A map which is topologically conjugate to a time-1 map of the form $T_{\mathbf{A}}$ or $\mathbf{R}^2$ for an induced nonlinear equation is said to be irreducible if the matrix $\mathbf{A}$ is one of the following four matrices:

$$\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}$$

These four time-1 maps are *prime* factors for nonlinear maps, in that each one can produce a simple nonlinear effect that in some sense cannot be made simpler. For this reason we call them the *irreducible fundamental Poincaré maps* or, simply, irreducible Poincaré maps.

Among the four irreducible maps in $\mathbf{R}^2$, only two are needed to describe all the irreducible nonlinear effects in $\mathbf{R}^2$ since the two diagonal maps are topologically conjugate to each other and the same is true for the two off-diagonal maps. Hence we have the following *minimal* irreducible set:

$$\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}$$

in $\mathbf{R}^2$. In $\mathbf{R}^3$ the situation is more complicated. To describe the general case we need two definitions from the theory of linear spaces:

Definition (Projection). A linear map $\mathbf{A}$ is called a projection if $\mathbf{A}^2 = \mathbf{A}$. Hence, the determinant of $\mathbf{A}$ is 1 or 0.

Definition (Nilpotent Map). A linear map $\mathbf{A}$ is called nilpotent if there exists an integer k such that $\mathbf{A}^k = \mathbf{0}$. Hence, the determinant of $\mathbf{A}$ is 0.

In the minimal set for $\mathbf{R}^2$ given above the first map is a projection and the second map is nilpotent. In three dimensions there are three things taken one, two and three at a time, or seven projections consisting of diagonal matrices having only 1's or 0's on the diagonal. Of these seven, only three are needed to include in a minimal set. These are

$$\begin{pmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix},$$

Among the nilpotent matrices we need only the following set:

$$\begin{pmatrix}0 & 1 & 0\\0 & 0 & 0\\0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 1 & 1\\0 & 0 & 0\\0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 1 & 1\\0 & 0 & 1\\0 & 0 & 0\end{pmatrix}$$
$$\begin{pmatrix}0 & 1 & 0\\0 & 0 & 1\\0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 1\\0 & 0 & 1\\0 & 0 & 0\end{pmatrix}$$

and thus the minimal irreducible set is 8.

Definition (Completely Reducible Maps). A map of $\mathbf{R}^n$ to $\mathbf{R}^n$ will be said to be *completely reducible* when it is topologically conjugate to a map which can be written as a *finite* composition of irreducible maps. We require that this factorization be a *global* factorization and not a *local* factorization.

We have the following lemma:

Lemma 25. *Every fundamental map is completely reducible.*

Proof. We need only prove the result for linear maps and the result must follow for the fundamental maps. The result follows for linear equations since these equations all have a set of constants of integration in any dimension. Using these constants of integration we can reduce the map to the form of Proposition 1 where the x coordinate is replaced by a vector and the functions $\Omega(x)$, $c(x)$, defined in Proposition 1 are real-valued functions of a vector in the remaining coordinates. Thus Proposition 1 applies. ■

Definition (Reducible Maps). A time-1 map will be called reducible if it can be approximated arbitrarily close on compact sets by a completely reducible map.

We now have a theorem for the local decomposition of all time-1 maps of autonomous ODEs in $\mathbf{R}^2$.

Theorem 1 (Locally Reducible Time-1 Maps in $\mathbf{R}^2$). *Let*

$$\dot{\mathbf{z}} = F(\mathbf{z})$$

where

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{z}$$

be an autonomous ODE in the plane which has a unique solution for each initial condition in $\mathbf{R}^2$.

Then the time-1 map for this ODE is locally completely reducible about each point $\mathbf{z}$ *where* $F(\mathbf{z})$ *is* C^1.

Proof. Since the ODE for $\mathbf{z}$ is autonomous, it can be written locally as a first-order ODE in phase-plane coordinates. Since every first-order ODE has an integrating factor [Ince, 1956] there exists a function on the integral curves which locally is constant on each integral curve. If we take this function as one coordinate and any system of orthogonal trajectories to the integral curves as the second coordinate, the ODE can be written as

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 \\ f(u, v) \end{pmatrix} .$$

Applying Proposition 1 we have a local time-t map given by

$$T_t \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ v \exp(\Omega(u)t) + c(u)(\exp(\Omega(u)t - 1)/\Omega(u) \end{pmatrix} ;$$

writing this map as

$$T_t \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ (v + c(u)/\Omega(u)) \exp(\Omega(u)t) - c(u)/\Omega(u) \end{pmatrix}$$

and taking $t = 1$ we see that it is the composition of three irreducible maps:

$$T_1 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ (v + c(u)/\Omega(u)) \end{pmatrix} ,$$

$$T_2 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ v \exp(\Omega(u)t) \end{pmatrix} ,$$

$$T_3 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ (v - c(u)/\Omega(u)) \end{pmatrix} .$$

Each of the above three maps is irreducible: T_1 is a twist, T_2 is a dilation/contraction map, and T_3 is the inverse of T_1 and is also a twist. ■

The next corollary has to do with equations having a complete set of integral invariants.

Corollary 5. *If*

$$\dot{\mathbf{z}} = F(\mathbf{z})$$

is an autonomous ODE in $\mathbf{R}^n$ *having a complete set of integral invariants, then the time-1 map is reducible.*

Proof. The key to the proof of Theorem 1 is the reduction of the equation to the form of Proposition 1. But this is the meaning of the assumption of the existence of a complete set of integral invariants. ■

The following lemmas follow from our definitions and the preceding lemmas:

Lemma 26. *The Chirikov map, the Hénon map, the gingerbread map, the twist-and-flip maps, and the twist-and-shift maps are completely reducible.*

4.2. *Inhomogeneous equations and the convolution theorem*

Corresponding to each matrix $\mathbf{A}$ is a natural periodically forced ODE defined by

$$\dot{\mathbf{z}} = f(\lambda(\mathbf{z} - \mathbf{s}(t)))\mathbf{A}(\mathbf{z}(t) - \mathbf{s}(t)) .$$

In the special case where

$$\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and

$$\mathbf{s}(t) = \begin{pmatrix} b(t) \\ 0 \end{pmatrix},$$

where $b(t) = \mathrm{sgn}(\sin(\omega t))$, we obtain the twist-and-flip ODE. On page 249 of our paper "Horseshoes in the Twist-and-Flip map," we defined a composition of "infinitesimal" twists for a sinusoidally forced nonlinear equation. We now show that there is an explicit expression for this infinite composition for the periodically forced fundamental map ODE given above. We prove this result in detail for the twist-and-flip ODE only; the proof is similar for the general case. The convolution expression we derive is an implicit, integral form, of the solution of the twist-and-flip ODE for an arbitrary forcing function $s(t)$. Since the solution of a periodically forced fundamental nonlinear ODE may be chaotic, we cannot expect an explicit expression for the convolution as is found in the linear case. However, since chaotic maps can be expressed recursively we can expect that our expression has a similar recursion. For continuous time recursion this means that the value of a solution of an ODE at time t^* can be expressed in terms of the values of the solution for all time $t < t^*$. Thus we have the following theorem in which the solution of a (possibly chaotic) forced ODE at time t is expressed in terms of a convolution formula that requires knowledge of the solution up to time t. In spite of this apparent limitation useful information can be obtained from this formula in special cases.

Theorem 2 (Twist-and-Flip Convolution). *Let*

$$\mathbf{z}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

be a solution of the ODE

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = r(t) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x - b(t) \\ y \end{pmatrix},$$

where $r(t) = \sqrt{(x - b(t))^2 + y^2}$.

Then the following is true:

(1)
$$\mathbf{z}(t) = \mathbf{s}(t) + \mathbf{A}\left(\int_0^t r(\eta)d\eta\right)(\mathbf{z}_0 - \mathbf{s}(0)) - \int_0^t \mathbf{A}\left(\int_\eta^t r(\xi)d\xi\right) ds(\eta).$$

(2) *If* $\mathbf{u}(t) \equiv \mathbf{z}(t) - \mathbf{s}(t)$, *then*

$$\|\mathbf{u}(t) + \int_0^t \mathbf{A}\left(\int_\eta^t \|\mathbf{u}(\xi)\|d\xi\right) ds(\eta)\| = \|\mathbf{A}\left(\int_0^t \|\mathbf{u}(\eta)\|d\eta\right)\mathbf{u}_0\| = \|\mathbf{u}_0\|.$$

(3) *If the function* $b(t) = b_0$ *is a constant then we have*

$$\|\mathbf{u}(t)\| = \|\mathbf{A}\left(\int_0^t \|\mathbf{u}(\eta)\|d\eta\right)\mathbf{u}_0\| = \|\mathbf{u}_0\|$$

and hence

$$\mathbf{u}(t) = \mathbf{A}(\mathbf{u}_0 t)(\mathbf{u}_0).$$

Proof. The proof is deferred to the appendix. Note that (2) is interesting in that it resembles an integral invariant.

By following the proof given in the appendix for the twist-and-flip convolution lemma we can greatly generalize this result. But first we need a technical lemma:

Lemma 27 (Technical Lemma). *Let* $g : \mathbf{R}^n \to \mathbf{R}$ *and assume that*

$$f(\exp(t\mathbf{A})\mathbf{z}_0) = f(\mathbf{z}_0)$$

for all t, *where* $\mathbf{A}$ *is a square matrix of dimension* m *and* $\mathbf{z}_0$ *is a vector of dimension* m.

Then

$$f(\exp(g(\mathbf{w})t\mathbf{A})\mathbf{z}_0) = f(\mathbf{z}_0)$$

for any $\mathbf{w}$ *in* $\mathbf{R}^n$.

Proof. The proof consists in observing that $g(\mathbf{w})$ is a scalar and so $g(\mathbf{w})t$ can be considered as just another time value, t_1. But for any time t_1 we have

$$f(\exp(t_1\mathbf{A})\mathbf{z}_0) = f(\mathbf{z}_0)$$

by our assumptions about f. ■

We now generalize the convolution theorem:

Theorem 3. *Let*

$$\mathbf{z}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

be the solution to

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = f(\lambda(\mathbf{z})) \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x - b(t) \\ y \end{pmatrix}$$

where $\lambda(u)$ is the function from Lemma 20 and f is an arbitrary continuous function of a single variable. Then,

(1)
$$\mathbf{z}(t) = \mathbf{s}(t) + \exp\left(\mathbf{A}\left(\int_0^t r(\eta)d\eta\right)\right)(\mathbf{z}_0 - \mathbf{s}(0)) - \int_0^t \exp\left(\mathbf{A}\left(\int_\eta^t r(\xi)d\xi\right)\right) d\mathbf{s}(\eta)$$

where $r(t) = f(\lambda(\mathbf{z}(t) - \mathbf{s}(t)))$.

(2) *Let $\mathbf{u}(t) = \mathbf{z}(t) - \mathbf{s}(t)$; then*

$$r\left(\mathbf{u}(t) + \int_0^t \exp\left(\mathbf{A}\left(\int_\eta^t r(\xi)d\xi\right)\right) d\mathbf{s}(\eta)\right) = r\left(\exp\left(\mathbf{A}\left(\int_0^t r(\eta)d\eta\right)\right)\mathbf{u}_0\right) = r(\mathbf{u}_0)\,.$$

Proof. Once item (1) is established by following the twist-and-flip case given in the appendix, item (2) follows from the technical lemma. ■

Note that the above results for nonlinear equations follow from the proofs for the linear case because the integral curves of the solutions of fundamental nonlinear equations agree with those of the solutions of related linear equations. It is this close connection between linear equations and fundamental nonlinear equations that makes a theory for these nonlinear equations possible. There is a parallel result for the general case corresponding to item (3) from the preceding twist-and-flip convolution lemma.

We may extend our results to the conventional forced oscillator by the following transformation:

Let

$$\dot{\mathbf{z}}(t) = f(\lambda(\mathbf{z}))\mathbf{A}\mathbf{z} - \dot{\mathbf{s}}(t)\,.$$

Then the transformation

$$\mathbf{w} = \mathbf{z} + \mathbf{s}(t)$$

reduces this equation to the form of the equations of convolution lemmas.

The derivation of the integral form of the solution of the twist-and-flip ODE when applied to linear forced ODEs is equivalent to the familiar convolution expression for the general solution of a linear forced ODE. In fact, it can be seen to be the same expression as occurs in the linear case by integrating the Stieltjes integral by parts and simplifying. We carry out this derivation in the appendix.

In the special case where $\mathbf{s}(t)$ is differentiable we can obtain a convolution theorem more easily. We need the following lemma, which is a restatement of Lemma 21.

Lemma 28. *Let $\Phi(t)$ be a matrix function of t. The function $\mathbf{z}(t)$,*

$$\mathbf{z}(t) = \exp(\Phi(t))\,,$$

is a solution of

$$\dot{\mathbf{z}} = \Phi(t)\mathbf{z}$$

if and only if the two matrices, $\Phi(t)$ and

$$\int_0^t \Phi(\eta)d\eta\,,$$

commute.

Proof. The proof is a direct computation.

As noted in Coddington & Levinson [1955], the above condition is satisfied in the case where $\Phi(t)$ is a constant matrix and a diagonal matrix. However, it is also satisfied when $\Phi(t) = f(t)\mathbf{A}$. The condition $\Phi(t) = f(t)\mathbf{A}$ is the condition that is used to prove the results about the fundamental nonlinear ODEs. Using this result we can derive a simplified convolution theorem. To do this we first prove some more technical lemmas.

Lemma 29. *Assume the ODE*

$$\dot{\mathbf{z}} = f(t)\mathbf{A}\mathbf{z}$$

has a unique solution for every initial condition, $\mathbf{z}_0$. Then

$$\mathbf{z}(t) = \exp(F(t)\mathbf{A})\mathbf{z}\,,$$

where

$$F(t) = \int_0^t f(\eta)d\eta\,.$$

Proof. The result follows from the preceding lemma.

Lemma 30. *Assume the ODE*

$$\dot{\mathbf{z}} = f(t)\mathbf{A}\mathbf{z} + \mathbf{s}(t)$$

has a unique solution for every initial condition, $\mathbf{z}_0$*.*

Then

$$\mathbf{z}(t) = \exp(F(t)\mathbf{A})\mathbf{z}_0 + \exp(F(t)\mathbf{A}) \times \left(\int_0^t \exp(-F(\eta)\mathbf{A})\mathbf{s}(\eta)d\eta\right),$$

where

$$F(t) = \int_0^t f(\eta)d\eta\,.$$

We have the following proposition:

Proposition 2. *Assume the ODE*

$$\dot{\mathbf{z}} = f(\mathbf{z}(t))\mathbf{A}\mathbf{z}(t) + \mathbf{s}(t)$$

has a unique solution for every initial condition, $\mathbf{z}_0$*.*

Then

$$\mathbf{z}(t) = \exp(F(t)\mathbf{A})\mathbf{z}_0 + \exp(F(t)\mathbf{A}) \times \left(\int_0^t \exp(-F(\eta)\mathbf{A})\mathbf{s}(\eta)d\eta\right),$$

where

$$F(t) = \int_0^t f(\mathbf{z}(\eta))d\eta\,.$$

Further, if λ *is an invariant function for the associated homogeneous linear ODE, then we have*

$$\lambda\Big(\mathbf{z}(t) - \exp(F(t)\mathbf{A}) \times \left(\int_0^t \exp(-F(\eta)\mathbf{A})\mathbf{s}(\eta)d\eta\right)\Big) = \lambda(\mathbf{z}_0)\,.$$

Proof. Let $\mathbf{z}(t)$ be the solution for which $\mathbf{z}(0) = \mathbf{z}_0$, and let $g(t) = f(\mathbf{z}(t))$, which is now a fixed function of t. Then

$$\dot{\mathbf{z}} = g(t)\mathbf{A}\mathbf{z} + \mathbf{s}(t)$$

and the previous lemma applies. The last line follows from the invariance of the function λ. ■

We now compute the Jacobian of the time-1 map for a general fundamental nonlinear ODE. We first prove some preliminary lemmas. The first lemma is an application of a theorem from Coddington & Levinson [1955]:

Lemma 31. *Let*

$$\dot{\mathbf{z}} = f(\mathbf{z})\mathbf{A}\mathbf{z}$$

have a unique solution for each initial condition, and let $D(\mathbf{z})$ *be the derivative of* $\mathbf{z}$ *with respect to the initial condition* $\mathbf{z}_0$*.*

Then

$$\det(D(\mathbf{z})) = \exp\left(\int_0^t \operatorname{tr}(D(f(\mathbf{z})\mathbf{A}\mathbf{z})(\eta)d\eta\right).$$

Proof. The proof is an application of Theorem 7.2, p. 25 of Coddington & Levinson [1955].

Lemma 32. *Given the assumptions of the above lemma we have*

$$D(f(\mathbf{z})\mathbf{A}\mathbf{z}) = \mathbf{A}\mathbf{z}\cdot\nabla f + f(\mathbf{z})\mathbf{A}\,.$$

Proof. The proof is a direct computation.

Lemma 33. *Given the assumptions of the preceding lemma we have the following trace formula:*

$$\operatorname{tr}(D(f(\mathbf{z})\mathbf{A}\mathbf{z})) = \nabla f\cdot\dot{\mathbf{z}}/f(\mathbf{z}) + f(\mathbf{z})\mathbf{A}\,.$$

Proof. The proof is a direct computation.

Corollary 6. *Given the results of the preceding lemmas we have the formula*

$$\det(D(\mathbf{z})) = \exp\left(f(\mathbf{z})\operatorname{tr}\mathbf{A}t + \int_0^t (\nabla f(\mathbf{z})\cdot\dot{\mathbf{z}}/f(\mathbf{z}))(\eta)d\eta\right).$$

Corollary 7. *Given the hypothesis of the above lemmas*

$$\operatorname{tr}(\mathbf{A}\mathbf{z}\cdot\nabla\lambda)(\mathbf{z}) = \tau(\nabla\lambda\cdot\dot{\mathbf{z}})/\lambda(\mathbf{z})\,.$$

We thus come to the final result, which is stated as a theorem because of its importance. We state the result in two dimensions but because of the results of the previous lemmas and corollaries it is true in n dimensions:

Theorem 4. *Let*

$$\mathbf{z}(t) = \exp(\lambda(\mathbf{z}_0)t\mathbf{A})\mathbf{z}_0$$

and consider the time-1 map $T_\tau(\mathbf{z}_0) = \mathbf{z}(\tau)$*; then the differential,* $D(T_\tau)$*, of* T_τ *with respect to* $\mathbf{z}_0$ *is*

$$\exp(\lambda(\mathbf{z}_0)\tau\mathbf{A}) + \mathbf{B}(\mathbf{z}_0, \tau),$$

where

$$\begin{aligned}\mathbf{B}(\mathbf{z}_0, \tau) &= \left[\left(\frac{\partial \exp(\lambda(\mathbf{z}_0)\tau\mathbf{A})}{\partial x_0}\mathbf{z}_0\right),\right.\\ &\quad \left.\times\left(\frac{\partial \exp(\lambda(\mathbf{z}_0)\tau\mathbf{A})}{\partial y_0}\mathbf{z}_0\right)\right]\\ &= \tau(\mathbf{A}\mathbf{z}_0 \cdot \nabla\lambda).\end{aligned}$$

The determinant of this differential is given by

$$\det(D(T_\tau)) = \exp(\lambda(\mathbf{z}_0)\tau\,\mathrm{tr}(\mathbf{A})).$$

If the trace of $\mathbf{A}$ *is zero both the linear and the nonlinear system defined by* $\mathbf{A}$ *are area-preserving.*

Proof. The proof follows from the previous results and the fact that $\nabla\lambda \cdot \dot{\mathbf{z}} = 0$.

Lemma 34. *Let* $\mathbf{z}$*,* T_τ*, and* $D(T_\tau)$ *be as in the above theorem. Then the trace of* $D(T_\tau)$ *is given by*

$$\begin{aligned}\mathrm{tr}(D(T_\tau)) &= \mathrm{tr}(\exp(\lambda(\mathbf{z}_0)\mathbf{A}\tau)) + (\nabla\lambda \cdot \dot{\mathbf{z}})/\lambda(\mathbf{z})\\ &= \mathrm{tr}(\exp(\lambda(\mathbf{z}_0)\mathbf{A}\tau)).\end{aligned}$$

Proof. The function λ is orthogonal to the integral curves determined by $\mathbf{z}$ and hence the second term is zero.

4.3. *Fundamental maps and topological conjugacy*

All time-1 maps of solutions of a fundamental nonlinear ODE are Poincaré maps by the factorization theory. Also, given two time-1 maps each of which arises from the solution of a different fundamental nonlinear ODE, their composition is a Poincaré map. The following lemma gives us conditions on the matrix $\mathbf{A}$ that generates a fundamental nonlinear ODE which ensures that the composition of two time-1 maps is topologically conjugate to its inverse. To do this we need the following definition familiar to students of the theory of Lie groups:

Lemma 35. *Let* $\mathbf{A}$*,* $\mathbf{B}$*,* $\mathbf{P}$ *be three matrices satisfying the relations*

$$\mathbf{AP} = -\mathbf{PA}$$

and

$$\mathbf{BP} = -\mathbf{PB}.$$

Define the time-1 maps

$$\begin{aligned}T_\mathbf{A}(\mathbf{z}) &= \exp(f(\lambda_\mathbf{A}(\mathbf{z}))\mathbf{A})\mathbf{z},\\ T_\mathbf{B}(\mathbf{z}) &= \exp(g(\lambda_\mathbf{B}(\mathbf{z}))\mathbf{B})\mathbf{z}.\end{aligned}$$

Also, assume that

$$f(\lambda_\mathbf{A}(\mathbf{P}(\mathbf{z}))) = f(\lambda_\mathbf{A}(\mathbf{z}))$$

and

$$g(\lambda_\mathbf{B}(\mathbf{P}(\mathbf{z}))) = g(\lambda_\mathbf{B}(\mathbf{z})).$$

Then $T_\mathbf{B} \circ T_\mathbf{A}$ *is topologically conjugate to* $(T_\mathbf{B} \circ T_\mathbf{A})^{-1}$ *with the topological conjugacy given by* $\mathbf{P} \circ T_\mathbf{B}^{-1}$.

Proof. We need only show that the maps $\mathbf{P}$, $T_\mathbf{A}$, and $T_\mathbf{B}$ satisfy the hypothesis of the conjugacy lemma, i.e. show that

$$\mathbf{P} \circ T_\mathbf{B} = T_\mathbf{B}^{-1} \circ \mathbf{P}$$

and

$$\mathbf{P} \circ T_\mathbf{A} = T_\mathbf{A}^{-1} \circ \mathbf{P}.$$

Showing the above formulas are true for maps requires showing they are true for each point $\mathbf{z}_0$, or, what is the same thing, showing they are true along integral curves where λ is a scalar constant. Therefore, the whole proof is reduced to the linear case. But the proof in the linear case is a direct result of using the exponential power series expression for these maps and the Lie bracket condition along with the additional fact that $h(\lambda_\mathbf{C}(\mathbf{P}(\mathbf{z}))) = h(\lambda_\mathbf{C}(\mathbf{z}))$ for $\mathbf{C} = \mathbf{A}$ or $\mathbf{B}$ and $h = f$ or g. We carry out this computation for one of the maps where we have dropped the scalar $h(\lambda_\mathbf{C}(\mathbf{z}))$ since due to the condition imposed above it cannot affect the computations:

$$\begin{aligned}\mathbf{P}\exp(\mathbf{A}) &= \mathbf{P}\left(I + \mathbf{A} + \frac{\mathbf{A}^2}{2} + \frac{\mathbf{A}^3}{6} + \cdots\right)\\ &= \mathbf{P}I + \mathbf{PA} + \frac{\mathbf{PA}^2}{2} + \frac{\mathbf{PA}^3}{6} + \cdots,\end{aligned}$$

which by the Lie bracket condition must equal

$$\mathbf{P}I - \mathbf{AP} + \frac{\mathbf{A}^2\mathbf{P}}{2} - \frac{\mathbf{A}^3\mathbf{P}}{6} + \cdots = \exp(-\mathbf{A})\mathbf{P}.$$

From this we may conclude that

$$\mathbf{P}T_{\mathbf{A}} = T_{\mathbf{A}}^{-1}\mathbf{P}\,.$$

A similar result holds for $T_{\mathbf{B}}$. Thus the conditions of the topological conjugacy lemma hold and we are done. ■

We note that the Chirikov map, the measure-preserving Hénon map, the twist-and-flip maps, the twist-and-shift maps, the gingerbread map, and many more maps are all examples of the composition of two time-1 maps from fundamental nonlinear vector fields for which the above lemma holds.

Lemma 36 (Reducible Poincaré Maps). *Consider the equation*

$$\ddot{x} + V(x) = f(t)\,, \tag{8}$$

where we assume the following:

(1) *Eq.* (8) *has a unique solution for each initial condition;*
(2) *The solutions of the unforced equation are elliptic, and* $V(-x) = -V(x)$*;*
(3) f *is periodic with period* p *and* $f(p/4 - t) = f(p/4+t)$*, which means that* f *has quarter-wave symmetry.*

Then the Poincaré map is reducible and is also topologically conjugate to its inverse about the vertical axis.

Proof. From Brown & Chua [1992], we know that the Poincaré map can be approximated pointwise by maps of the form

$$\Phi = T_{a_1} \circ T_{a_2} \circ T_{a_3} \circ \cdots \circ T_{a_n} \circ T_{a_{n-1}} \circ \cdots T_{a_2} \circ T_{a_1}\,.$$

This follows from the condition of quarter-wave symmetry in the function f. Also, since $V(-x) = -V(x)$ we have $\mathbf{P} \circ T_{a_i} = T_{a_i}^{-1} \circ \mathbf{P}$, where $\mathbf{P}$ is a reflection about the horizontal axis. Next we know that each T_{a_i} is a time-1 map for the autonomous elliptic ODE

$$\ddot{x} + V(x) = f(t_i)$$

where $a_i = f(t_i)$. [This means that the maps T_{a_i} are elliptic twists about the center (a_i, 0.0).] Since this equation is elliptic, there is a change of coordinates for which it has the form

$$\dot{I} = 0\,,$$
$$\dot{\theta} = F(I)\,,$$

which is the equation for an irreducible map in action-angle coordinates. Thus each T_{a_i} is irreducible.

Since $\mathbf{P} \circ T_{a_i} = T_{a_i}^{-1} \circ \mathbf{P}$ for each i, $\mathbf{P} \circ \Phi = \Phi^{-1} \circ \mathbf{P}$ by Lemma 11 and so by Lemma 10, the Poincaré map for Eq. (8) is arbitrarily close to a map that is topologically conjugate to its inverse. ■

To generalize our results concerning fundamental and irreducible maps in $\mathbf{R}^n$ we draw on the theory of the linear first-order partial differential equation (PDE). This is because the solutions of a first-order ODE for a vector field $\dot{\mathbf{z}}$ are connected to the solutions of a first order linear PDE for a function $f(\mathbf{z})$, by the relation

$$\nabla f \cdot \dot{\mathbf{z}} = 0\,,$$

where ∇f is the gradient of f. The higher order cases simply involve more arbitrary functions λ_1, $\lambda_2, \ldots, \lambda_{n-1}$ and thus there are many more fundamental maps in higher-dimensional spaces. As we noted earlier, there are eight irreducible maps that make up a minimal set in three-dimensional space. In general there are N-irreducible, nonconjugate projections in N-dimensional space. The nonconjugate nilpotent maps are more complicated to enumerate.

4.4. *Classes of constructive Poincaré maps*

In Brown & Chua [1991] we set forth five classes of equations for which the Poincaré map could be obtained in closed form. We now update that classification.

Class 1 (Completely Reducible Poincaré Maps). This is the class of Poincaré maps which are completely reducible by our definition. This means that they can be factored into a finite number of irreducible components and this factorization does not vary over all of $\mathbf{R}^n$. Often this will be an action-angle coordinate system. As an example an equation of the form

$$\begin{pmatrix} I \\ \theta \end{pmatrix} = \begin{pmatrix} I_0 \\ \theta_0 + \Omega(I_0)t \end{pmatrix}$$

is the solution of a fundamental nonlinear ODE and its time-1 map is irreducible. This new *class 1* includes all of classes 1–5 from Brown & Chua [1991] and is divided into two subclasses depending on whether the matrix $\mathbf{A}$ of the time-1 map $T_{\mathbf{A}}$ has trace zero or nonzero. In subclass 1(a) we require the trace of $\mathbf{A}$ to be zero for every factor of the completely reducible map.

Class 1(b) (Dissipative Poincaré Maps). This class is defined by Poincaré maps where the determinant of the Jacobian derivative of the map is not equal to 1. Class 1(b) includes all dissipative Poincaré maps from Brown & Chua [1991].

The Chirikov map and the measure-preserving Hénon map are of class 1(a) and the general Hénon map and the one-dimensional embedded maps are of class 1(b).

Class 2. This class consists of the reducible Poincaré maps and is also divided into two subclasses, one of which is measure-preserving.

5. Irreducible Maps and Homoclinic Manifolds

In this section we present some conditions for the existence of hyperbolic fixed points and homoclinic manifolds for maps that are the composition of two irreducible maps.

Lemma 37. *Let FT be a map where F and T are irreducible maps. Then a necessary condition for the map FT to have a hyperbolic fixed point is that the integral curves of the flows for F and T be transversal at a fixed point.*

This may also be stated as follows: a necessary condition for FT to have a hyperbolic fixed point is that the flow for F does not preserve the integral curves of T at the fixed point.

We have the following necessary and sufficient condition for the existence of a homoclinic manifold:

Theorem 5 (Homoclinic Manifold Theorem). *Let F_1 and F_2 be two diffeomorphisms of $\mathbf{R}^2$ such that $F_1^2(x) = F_2^2(x) = x$ for all $x \in \mathbf{R}^2$. Let $p \in \mathbf{R}^2$ be such that $F_1(p) = F_2(p) = p$. Let $\Phi = F_1 \circ F_2$ and assume that p is a hyperbolic fixed point of Φ.*

Then $W^u(p) = W^s(p)$ iff the following things hold:

1. *There is a point q_a which is a fixed point of F_1 but which is not a fixed point of F_2, and there is a point q_b which is a fixed point of F_2 but which is not a fixed point of F_1 such that q_a, $q_b \in W^u(p)$;*
2. *No other fixed points of F_1 or F_2 are in $W^u(p) - \{p\}$;*
3. *If $W^u(p)$ is parametrized by arc length then $q_b < q_a$.*

Proof. See Fig. 11. If there exist more fixed points than those specified in the hypothesis, then the unstable manifold must cross the stable manifold at such a point by the symmetry imposed by the involution conditions and hence the homoclinic manifold would not be a simple closed curve in $\mathbf{R}^2$. ■

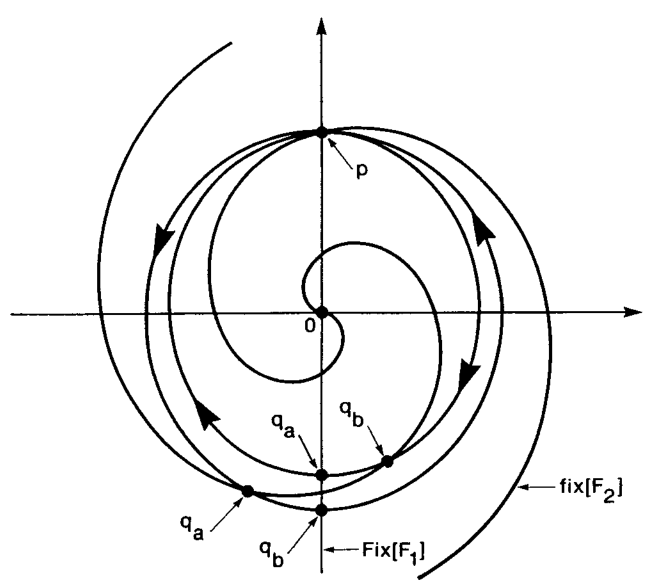

Fig. 11. The unstable manifold has two loops, one inside the other. The vertical axis is fixed by F_1 and the indicated spiral is fixed by F_2. By following the order indicated by the direction of expansion of the unstable manifold, we can see that the two fixed points are ordered. There is a pair of fixed points for each of the two loops of the unstable manifold. A third fixed point is not possible without reversing the order of expansion or creating a horseshoe.

The following maps are covered by this lemma:

Example

(1) Let $F_1 = R$ and $F_2 = PT$, where R is the reflection about the vertical axis, P is the reflection about the horizontal axis, and T is any twist map. $F_1 \circ F_2 = FT$, the twist-and-flip map.
(2) Let $F_1 = LP$ and $F_2 = PT$, where L is any odd shift map (recall that an odd shift map is defined as a shift map with shift function f such that $f(-x) = -f(x)$). Then $F_1 \circ F_2 = LT$. The Chirikov/standard map is a special case of this example.
(3) Let $F_1 = LR$ and $F_2 = RT$, where L is any even shift map. Then $F_1 \circ F_2 = LT$. This is an extension of the Chirikov map.
(4) If we consider the above example in rectangular coordinates, then the Hénon map with $b = -1$ is a special case. When $b = +1$, the diffeomorphisms arising from the class of one-dimensional maps are covered by these theorems.
(5) Poincaré maps arising from ODEs covered by the factorization lemmas are covered by this theorem.
(6) Let $F_1 = AS$ and $F_2 = ST$, where A is any linear hyperbolic map. Then there exists an isometric involution S such that $A^{-1}S = SA$ and $T^{-1}S = ST$. In this case $F_1 \circ F_2 = AT$. This is not contained in the previous examples.
(7) Consider the following ODE:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ -g(x) \end{pmatrix},$$

where $g(-x) = g(x)$. We seek a one-step integrator which is optimal in the following sense: $\det(D(\Phi_h(X)) = 1$ for all step sizes h, since this is true for the time-t map for the ODE; $\Phi_h \sim (\Phi_h)^{-1}$, this is also true for the ODE; and, $D_t(\Phi_t(X))$ evaluated at 0 is given by the above ODE. By direct calculation we can confirm that

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0 + ty(t) \\ y_0 - tg(x_0) \end{pmatrix}$$

has all of these properties. In particular,

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} y - t(y - tg(x - ty)) \\ -g(x - ty) \end{pmatrix}$$

so that for $t = 0$ we have

$$\begin{pmatrix} \dot{x}(0) \\ \dot{y}(0) \end{pmatrix} = \begin{pmatrix} y_0 \\ -g(x_0) \end{pmatrix}.$$

Since this equation can be factored as a composition of a twist map and a shift map, it is measure-preserving; and since the shift is odd, we know that every time-t map is topologically conjugate to its inverse. For small h this one-step integrator is stable. For any step size, this integrator is a diffeomorphism which is a twist-and-shift map. This example draws a close connection between twist-and-shift maps and integrators. If we choose $g(x) = x^3$ and treat h as a parameter, we see that the integrator becomes unstable as a result of the presence of horseshoes, or chaos.

6. Engineering Theory of Nonlinear Effects

We conclude this paper with a suggestion of an "engineering" theory of nonlinear phenomena that parallels in some interesting repects Hamiltonian theory.

The mathematical theory presented in the previous sections suggests that, in a practical sense, there exist only a "small" number of "simple" nonlinear "effects." Examples are the twist, the shift, and the nonlinear dilation and contraction. If this is true then it is reasonable to expect that most nonlinear dynamical systems of interest can be built up from these simple effects by composition and coordinate transformations. A good example of this is that given a Hamiltonian system in the plane having a fixed point of the center type, there always exists a coordinate transformation in which the dynamics are given by a twist map.

This engineering theory justifies, as far as modeling and design of nonlinear systems having specified features is concerned, a search for maps which are composed of simple components (mentioned above), combined with the appropriate coordinate transformations. It also suggests that in many cases approximations to the coordinate transformations may provide a map having the same dynamical properties as a given map although not being "numerically" as accurate as numerical integration by conventional means.

It is thus reasonable to assume from an engineering perspective that only a finite number of nonlinear transformations are needed to construct most nonlinear maps of interest. Four of these maps, discussed above, considered in action-angle coordinates, are the twist

$$\begin{pmatrix} I = I_0 \\ \theta = \theta_0 + \Omega(I)t \end{pmatrix},$$

The shift, which is the transpose of the twist, and the two nonlinear contraction/dilation maps given by the equation,

$$\begin{pmatrix} I = I_0 \exp(\alpha t g(\theta)) \\ \theta = \theta_0 \end{pmatrix}.$$

For example, simple limit cycles in action-angle coordinates can be constructed from limit cycles defined by the four fundamental maps in conjunction with coordinate transformations. Limit cycle equations in action-angle variables such as the one given by the map

$$\begin{pmatrix} I \\ \theta \end{pmatrix} = \begin{pmatrix} \dfrac{bI_0}{(I_0 + (b - I_0)\exp(-bt))} \\ \theta_0 - I_0 + \Omega(I) \end{pmatrix}$$

are a coordinate transformation of a linear limit cycle equation since it is a solution of Bernoulli's equation. The solution to the linear limit cycle equation

$$\dot{x} + x = b$$

is a composition made up from the four irreducible maps.

The Differential Equations for Coordinate Transformations in Two Dimensions

If we are given a vector field,

$$\begin{pmatrix} \dot{x} = f(x, y) \\ \dot{y} = g(x, y) \end{pmatrix}.$$

Suppose that there exists a coordinate transformation

$$\begin{pmatrix} I \\ \theta \end{pmatrix} = \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix}$$

to some action-angle coordinate system such that the key dynamical properties are preserved (limit cycles, for example). If the action-angle variables also have a limit cycle, but a much simpler one, then an approximation to the necessary coordinate transformation may be found that is very useful.

The actual PDE can be written down as follows. Suppose, for example, that the action-angle variables satisfy the equation

$$\begin{pmatrix} \dot{I} \\ \dot{\theta} \end{pmatrix} = G\begin{pmatrix} I \\ \theta \end{pmatrix}.$$

Then we have the following PDE for the coordinate transformation functions u, v:

$$\begin{pmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{pmatrix} \begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix} = G\begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix}.$$

In this equation everything is known except the functions u, v. In practice, this equation is no easier to solve than the original system of ODEs since its solution requires solving the original ODEs, but if through geometric reasoning we are able to choose G to have the essential properties of the vector field of interest and to be such that in the action-angle coordinates a closed form solution can be found, then a dynamical approximation to the solution may be found for the equation we are interested in that retains the important dynamics. The effect of doing all of this is to obtain a dynamical approximation to a map as opposed to a numerical approximation.

The PDE given above has a solution involving two arbitrary functions. These functions are specified in the Hamiltonian case by requiring that the transformation have determinant 1, and that the action-angle system also be Hamiltonian. In the non-Hamiltonian case, i.e., where we are only seeking a dynamical approximation, we may relax these two constraints in order to obtain other advantages. For example, the function G may be chosen to be any convenient function. The measure-preserving character of the transformation may be relaxed. Or the approximation need not be exact.

In order to illustrate what can be done in a practical setting, we revisit the Van der Pol equation.

Example (Dynamical Synthesis of a Global Solution to the Van Der Pol Equation). Consider the equation

$$\ddot{x} + \phi(x)\dot{x} + x = 0.$$

In an earlier section we derived an action-angle formulation of an autonomous system having a limit cycle. This system was given by the ODE:

$$\dot{r} = \beta r(b - r),$$
$$\dot{\theta} = 1,$$

or, equivalently,

$$\dot{r} = br(1 - ar)\,, \tag{9}$$

$$\dot{\theta} = 1\,. \tag{10}$$

Our theory suggests that it must be possible to transform Eq. (10) to a coordinate system in which it models the Van der Pol equation. Further, we will try to find a transformation composed of the four irreducible maps. First we transform the Van der Pol equation to a first-order system:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ -x - \phi(x)y \end{pmatrix}.$$

We will now seek to find a twist of an action-angle system that is a dynamical approximation to this system. We assume that the action-angle system looks like

$$\begin{pmatrix} \dot{I} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} h(I) \\ \Omega(I) \end{pmatrix}.$$

In rectangular coordinates this system is given by

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \frac{\partial \theta}{\partial v} & -\frac{\partial I}{\partial v} \\ -\frac{\partial \theta}{\partial u} & \frac{\partial I}{\partial u} \end{pmatrix} \begin{pmatrix} h(I) \\ \Omega(I) \end{pmatrix}$$

where as $t \to \infty$, $\dot{I} \to 0$ and the system approaches a Hamiltonian system.

We select as the change of coordinates, from the u, v rectangular version of the limit cycle system to the approximating system, a twist:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -f & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix},$$

where the function f is to be determined by some criteria, and we have

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} \to \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

as $t \to \infty$. Making all substitutions and taking the limits as $t \to \infty$ we obtain the following equations for the approximation:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y + f(x) \\ -x - f'(x)(y + f(x)) \end{pmatrix}.$$

We require that this twisted vector field fit the Van der Pol vector field in some average, min–max or other sense as may be of practical value. One way to do this is to subtract the Van der Pol vector field from this vector field and form the following expression for the square of their differences:

$$\Delta = (f(x))^2 + ((f'(x) - \phi(x))y + (f(x)f'(x)))^2\,.$$

Now the twist function f can be chosen in many ways. But choosing $f' = \phi$ is tempting since this would remove one of the terms from Δ. After having done this our twist transformation looks like

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\Phi & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

where $\Phi'(x) = \phi(x)$. Clearly this transformation cannot change the x-coordinate. But upon an examination of the y-coordinate we see that it looks a lot like the derivative of the solution to the Van der Pol equation. Hence we select the integral of this function as our x-coordinate. We summarize all of this by the transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & g \\ -\Phi & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

where $g'(v) = f(u)/u$ on the limit cycle determined by the rectangular action angle coordinate system defined by $u = (1/a)\cos(-t+\theta_0)$, $v = (1/a) \times \sin(-t + \theta_0)$ on the limit cycle. For the Van der Pol equation with $\varepsilon = 1.0$, $\Phi(x) = x^3/3 - x$ and so $g(v) = (1/(3a^2) - 1)v - v^3/9$. Taking $a = 0.55$ we obtain the limit cycle seen in Fig. 12.

In Fig. 13, we show the time solution obtained from this equation for the functions $x(t)$ and $y(t)$.

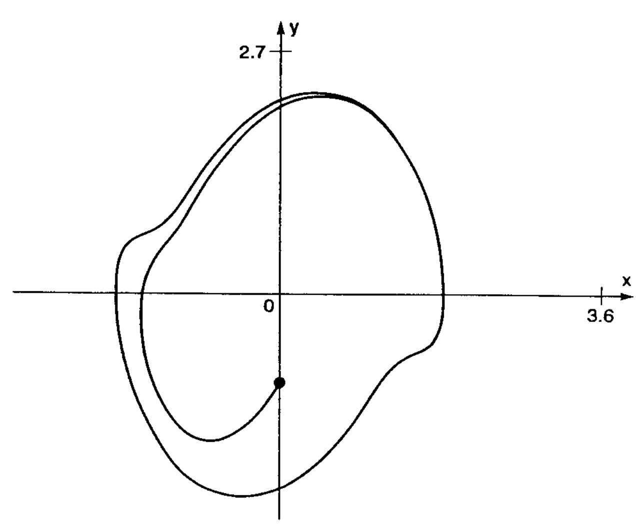

Fig. 12. The phase plane portrait of the dynamical synthesis solution of the Van der Pol equation with $\varepsilon = 1.0$. The fitting parameter a has the value 0.55.

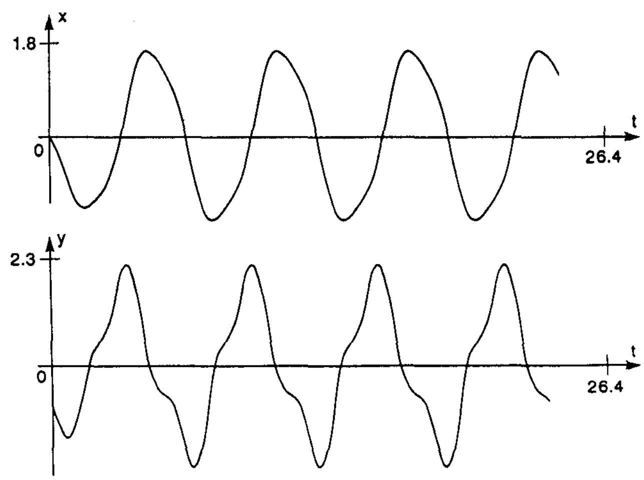

Fig. 13. The time series for the dynamical synthesis solution of the Van der Pol equation with $\varepsilon = 1.0$. The fitting parameter a has the value 0.55. The top figure is the time series derived by dynamical synthesis. The bottom figure is the time series obtained by integration of the equation used to produce the time series in the top figure.

For the Van der Pol equation, this coordinate transformation is invertible on the limit cycle since $1 + f'(u)f(u)/u \neq 0$ there.

Our approach can be thought of as a refinement of the method of Kyrlov & Bogoliubov (see Ross [1964], page 508), for a good elementary treatment, in that we begin with what is "essentially" their first approximation and assume that a better approximation can be found by the process of twisting and shifting their first approximation. Using the example of the Hamiltonian system, we conjecture there exists some practical twist-and-shift coordinate transformation that, for the most part, accounts for the nonlinear form in which the initial conditions occur in nonlinear systems.

References

Arrowsmith, D. K. & Place, G. M. [1990] *An Introduction to Dynamical Systems* (Cambridge University Press, Cambridge).

Bowman, F. H. [1961] *Introduction to Elliptic Functions with Applications* (Dover, New York).

Brown, R. [1992] "Generalizations of the Chua equations," *Int. J. Bifurcation and Chaos* **2**(4), 889–909.

Brown, R. & Chua, L. [1991] "Generalizing the twist-and-flip paradigm," *Int. J. Bifurcation and Chaos* **1**(2), 385–416.

Brown, R. & Chua, L. [1992] "Generalizing the twist-and-flip paradigm II," *Int. J. Bifurcation and Chaos* **2**(1), 183–191.

Brown, R. & Chua, L. [1992] "Chaos or turbulence," *Int. J. Bifurcation and Chaos* **2**(4), 1005–1009.

Chua, L. [1980] "Device modeling via basic nonlinear circuit elements," *IEEE Trans. Circuits and Syst.* **27**, 1014–1044.

Chua, L. [1980] "Dynamical nonlinear networks: State-of-the-art," *IEEE Trans. Circuits and Syst.* **27**, 1059–1087.

Chua, L., Brown, R. & Hamilton, N. [1993] "Fractals in the twist-and-flip circuit," *Proceedings of the IEEE*, Special Issue, Fall 1993.

Coddington, E. & Levinson, N. [1955] *Ordinary Differential Equations* (McGraw-Hill, New York).

Guckenheimer, J. & Holmes, P. [1983] *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields* (Springer-Verlag, New York).

Hartman, P. [1982] *Ordinary Differential Equations* (Birkhauser, Boston), 2nd edition.

Parker, T. & Chua, L. [1989] *Practical Numerical Algorithms for Chaotic Systems* (Springer-Verlag, Berlin).

Thompson, J. & Stewart, H. [1986] *Nonlinear Dynamics and Chaos* (John Wiley, New York).

Davis, Harold, T. [1960] *Introduction to Nonlinear Differential and Integral Equations* (Dover, New York).

Devaney, Robert, L. [1984] "A piecewise linear model for the zones of instability of an area-preserving map," *Physica* **D10**, 387–393.

Hénon, M. [1969] "Numerical study of quadratic area-preserving mappings," *Quarterly of Applied Mathematics* **XXVII**(3).

Hille, Einar, [1969] *Lectures on Ordinary Differential Equations* (Addison-Wesley Publishing Company, Menlo Park, Calif.).

Ince, E. I. [1956] *Ordinary Differential Equations* (Dover Publications, New York).

Sneddon, I. N. [1957] *Elements of Partial Differential Equations* (McGraw-Hill, New York).

Appendix
Proof of Lemmas

We prove the four lemmas of Proposition 1 in order:

Lemma 13

$$f(x, y, t) = \int_0^y \exp(tg(x, \eta, t))d\eta + a\int_0^t \exp(\eta g(x, 0, \eta))d\eta\,.$$

Proof. First observe that

$$f(x, y, t) = \int_0^y \exp(g(x, \eta, t))d\eta + h(x, t)$$

by partial integration with respect to y. By the group property $f(x, y, 0) = y$ for all y; hence we have $y = \int_0^y \exp(g(x, \eta, 0))d\eta + h(x, 0)$. Differentiating this expression with respect to y we obtain $\exp(g(x, y, 0)) = 1$ for all x, y. Hence, $g(x, y, 0) = 0$ for all x, y. This last result means that $g(x, y, t) = tk(x, y, t)$ for some function k. We will therefore rewrite f (using the letter g in place of the letter k for convenience) as follows:

$$f(x, y, t) = \int_0^y \exp(tg(x, \eta, t))d\eta + h(x, t).$$

From this we conclude that $h(x, 0) = 0$ for all x.

Now let

$$T_t\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y_1 \end{pmatrix}.$$

Hence

$$y_1 = \int_0^y \exp(tg(x, \eta, t))d\eta + h(x, t).$$

Using the group property $T_{s+t} = T_s \circ T_t$ we have

$$T_s\begin{pmatrix} x \\ y_1 \end{pmatrix} = \begin{pmatrix} x \\ y_2 \end{pmatrix},$$

where

$$\begin{aligned} y_2 &= \int_0^{y_1} \exp(sg(x, \eta, s))d\eta + h(x, s) \\ &= \int_0^y \exp((s+t)g(x, \eta, s+t))d\eta + h(x, s+t). \end{aligned}$$

For $y = 0$, $y_1 = h(x, t)$ so that

$$\int_0^{h(x,t)} \exp(sg(x, \eta, s))d\eta + h(x, s) = h(x, s+t).$$

Differentiating this expression with respect to t we get

$$\exp(sg(x, h(x, t)), s))h'(x, t) = h'(x, s+t),$$

where we have omitted the partial differentiation notation since x is always a constant throughout all proofs. We know that $h(x, 0) = 0$, and for $t = 0$ we have the equation

$$\exp(sg(x, 0, s))h'(x, 0) = h'(x, s).$$

This is a differential equation for h from which we conclude that

$$h(x, t) = a\int_0^t \exp(\eta g(x, 0, \eta))d\eta,$$

where $a = h'(x, 0)$, and so

$$f(x, y, t) = \int_0^y \exp(tg(x, \eta, t))d\eta + a\int_0^t \exp(\eta g(x, 0, \eta))d\eta. \quad \blacksquare$$

Lemma 14. *For all a, s, x, y we have*

$$\begin{aligned} &a \exp(sg(x, y, s)) \\ &\quad = a\exp(s, g(x, 0, s)) + \int_0^y \exp(sg(x, \eta, s)) \\ &\quad\quad \times \left(g(x, \eta, s) + s\frac{\partial g(x, \eta, s)}{\partial t}\right)d\eta \end{aligned}$$

Proof. From the result of Lemma 2 we apply the group property again to obtain

$$\begin{aligned} &\int_0^{y_1} \exp(sg(x, \eta, s))d\eta + a\int_0^s \exp(\eta g(x, 0, \eta)) \\ &\quad = \int_0^y \exp((s+t)g(x, \eta, s+t))d\eta \\ &\quad\quad + a\int_0^{s+t} \exp(\eta g(x, 0, \eta))d\eta. \end{aligned}$$

Since

$$h(s+t) = h(s) + \int_0^{h(t)} \exp(sg(x, \eta, s)d\eta$$

this expression simplifies to

$$\begin{aligned} &\int_0^{y_1} \exp(sg(x, \eta, s))d\eta \\ &\quad = \int_0^y \exp((s+t)g(x, \eta, s+t))d\eta \\ &\quad\quad + \int_0^{h(t)} \exp(s, g(x, \eta, s))d\eta. \end{aligned}$$

Differentiating this result with respect to t we get

$$\begin{aligned} &\exp(sg(x, y_1, s))\dot{y}_1 \\ &\quad = \int_0^y \exp((s+t)g(x, \eta, s+t)) \\ &\quad\quad \times \left(g(x, \eta, s+t) + (s+t)\frac{\partial g(x, \eta, s+t)}{\partial t}\right)d\eta \\ &\quad\quad + \exp(s, g(x, h(x, t))\dot{h}; \end{aligned}$$

for $t = 0$ we have

$$a\ \exp(sg(x,\ y,\ s)) = a\ \exp(s,\ g(x,\ 0,\ s)) + \int_0^y \exp(sg(x,\ \eta,\ s)) \times \left(g(x,\ \eta,\ s) + s\frac{\partial g(x,\ \eta,\ s)}{\partial t}\right)d\eta\,. \quad \blacksquare$$

Lemma 15

$$g(x,\ y,\ t) = F(x,\ at + y)\,.$$

Proof. Beginning with the result in Lemma 13 and differentiating this expression with respect to y, we get

$$sa\ \exp(sg(x,\ y,\ s))g_y(x,\ y,\ s) = \exp(sg(x,\ y,\ s))\left(g(x,\ y,\ s) + s\frac{\partial g(x,\ y,\ s)}{\partial t}\right),$$

which may be rewritten as

$$tag_y(x,\ y,\ s) = g(x,\ y,\ s) + t\frac{\partial g(x,\ y,\ t)}{\partial t}$$

or

$$tag_y = g + tg_t\,.$$

The general solution of this partial differential equation is given by

$$g(x,\ y,\ t) = \frac{c}{t} + F(x,\ at + y)\,,$$

where F is an arbitrary function and c is an arbitrary constant. Since we require g to be differentiable everywhere in all variables, we must take $c = 0$. $\blacksquare$

Lemma 16

$$F(x,\ u) = \Omega(x)\,.$$

Proof. Substituting this information into the equation for $f(x,\ y,\ t)$ and applying the group property with $s = -t$, we get the functional equation

$$F(x,\ at + y) = F(x,\ y_1 - at)\,.$$

If F is not a constant with respect to the second variable then we can differentiate this equation with respect to the second variable and divide out the derivative to get

$$a\frac{\partial y_1}{\partial y} = \dot{y}_1 - a\,.$$

We know

$$\frac{\partial y_1}{\partial y} = \exp(tF(x,\ at + y))$$

for all t so that

$$\dot{y}_1 = a\ \exp(tF(x,\ at{+}y)){+}\int_0^y \exp(tF(x,\ at{+}\eta))d\eta\,;$$

from this we conclude that

$$a + a\ \exp(tF(x,\ at + y)) = a\ \exp(tF(x,\ at + y)) + \int_0^y \exp(tF(x,\ at + \eta))F(at + \eta)d\eta$$

or

$$a = \int_0^y \exp(tF(x,\ at + \eta))F(x,\ at + \eta)d\eta$$

for all t, implying that $F(x,\ u) = 0$ for all u. But this is a contradiction, and hence $F(x,\ u) = \Omega(x)$. $\blacksquare$

Theorem 2. *Let*

$$\mathbf{z}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

be a solution of the ODE

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = r(t)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x - b(t) \\ y \end{pmatrix},$$

where $r(t) = \sqrt{(x - b(t))^2 + y^2}$.

Then the following is true:

(1)
$$\mathbf{z}(t) = \mathbf{s}(t) + \mathbf{A}\left(\int_0^t r(\eta)d\eta\right)(\mathbf{z}_0 - \mathbf{s}(0)) - \int_0^t \mathbf{A}\left(\int_\eta^t r(\xi)d\xi\right)d\mathbf{s}(\eta)\,.$$

(2) *If* $\mathbf{u}(t) \equiv \mathbf{z}(t) - \mathbf{s}(t)$, *then*

$$\|\mathbf{u}(t) + \int_0^t \mathbf{A}\left(\int_\eta^t \|\mathbf{u}(\xi)\|d\xi\right)d\mathbf{s}(\eta)\| = \|\mathbf{A}\left(\int_0^t \|\mathbf{u}(\eta)\|d\eta\right)\mathbf{u}_0\| = \|\mathbf{u}_0\|\,.$$

(3) *If the function* $b(t) = b_0$ *is a constant then we have*

$$\|\mathbf{u}(t)\| = \|\mathbf{A}\left(\int_0^t \|\mathbf{u}(\eta)\|d\eta\right)\mathbf{u}_0\| = \|\mathbf{u}_0\|$$

and hence

$$\mathbf{u}(t) = \mathbf{A}(\mathbf{u}_0 t)(\mathbf{u}_0)\,.$$

Proof. By geometric considerations we can see that at a fixed time t_0 and over a small time increment h, the forced twist equation must be described approximately by

$$\mathbf{z}(t_0 + h) \cong A(hr(t_0))(\mathbf{z}(t_0) - \mathbf{s}(t_0)) + \mathbf{s}(t_0)\,,$$

where

$$\mathbf{z}(t) = \begin{pmatrix} x \\ y \end{pmatrix}$$

and

$$\mathbf{s}(t) = \begin{pmatrix} b(t) \\ 0 \end{pmatrix}\,.$$

Also, $A(u)$ occurring in the above and later expressions is the matrix

$$A(u) = \begin{pmatrix} \cos(u) & -\sin(u) \\ \sin(u) & \cos(u) \end{pmatrix}\,.$$

This is the solution of the twist-and-flip ODE

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = r(t) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x - a \\ y \end{pmatrix}\,,$$

where $a = b(0)$ and $r(t) = \sqrt{(x-a)^2 + y^2}$ over a short time interval, h. If we iterate this expression starting at an initial point $\mathbf{z}_0$, we have

$$\mathbf{z}_1 = A(hr_0)(\mathbf{z}_0 - \mathbf{s}(0)) + \mathbf{s}(0)\,,$$
$$\mathbf{z}_2 = A(hr_1)(\mathbf{z}_1 - \mathbf{s}(h)) + \mathbf{s}(h)\,,$$
$$\mathbf{z}_3 = A(hr_2)(\mathbf{z}_2 - \mathbf{s}(2h)) + \mathbf{s}(2h)\,,$$

where $r_i = \|\mathbf{z}_i - \mathbf{s}(ih)\|$. In general,

$$\mathbf{z}_{k+1} = A(hr_k)(\mathbf{z}_k - \mathbf{s}(kh)) + \mathbf{s}(kh)$$

so that the composition is given by

$$\begin{aligned}\mathbf{z}_{N+1} = {} & A(hr_N)A(hr_{N-1})A(hr_{N-2}) \\ & \cdots A(hr_0)(\mathbf{z}_0 - \mathbf{s}(0)) \\ & + \sum_{k=1}^{N} A(hr_N)A(hr_{N-1})A(hr_{N-2}) \\ & \cdots A(hr_{N-k})(b((k-1)h) - b(kh)) + b(Nh) \\ = {} & A\left(h \sum_{i=0}^{N} r_i\right)(\mathbf{z}_0 - \mathbf{s}(0)) + \sum_{k=1}^{N} A\left(h \sum_{i=0}^{k} r_{N-i}\right) \\ & \times (\mathbf{s}((k-1)h) - \mathbf{s}(kh)) + \mathbf{s}(Nh)\,;\end{aligned}$$

as $h \to 0$, this expression becomes

$$\begin{aligned}\mathbf{z}(t) = {} & \mathbf{s}(t) + A\left(\int_0^t r(s)ds\right)(\mathbf{z}_0 - \mathbf{s}(0)) \\ & - \int_0^t A\left(\int_s^t r(\xi)d\xi\right)d\mathbf{s}(s)\,.\end{aligned}$$

This must be considered a Stieltjes integral when $\mathbf{s}(t)$ is not differentiable.

Now make a change of variable $\mathbf{u}(t) = \mathbf{z}(t) - \mathbf{s}(t)$. Then we have

$$\begin{aligned}\mathbf{u}(t) = {} & A\left(\int_0^t \|\mathbf{u}(s)\|ds\right)\mathbf{u}_0 \\ & - \int_0^t A\left(\int_s^t \|\mathbf{u}(\xi)\|d\xi\right)d\mathbf{s}(s)\end{aligned}$$

or

$$\begin{aligned}\mathbf{u}(t) + \int_0^t A\left(\int_s^t \|\mathbf{u}(\xi)\|d\xi\right)d\mathbf{s}(s) \\ = A\left(\int_0^t \|\mathbf{u}(s)\|ds\right)\mathbf{u}_0\end{aligned}$$

so that,

$$\begin{aligned}&\left\|\mathbf{u}(t) + \int_0^t A\left(\int_s^t \|\mathbf{u}(\xi)\|d\xi\right)d\mathbf{s}(s)\right\| \\ &\qquad = \left\|A\left(\int_0^t \|\mathbf{u}(s)\|ds\right)\mathbf{u}_0\right\| = \|\mathbf{u}_0\|\,.\end{aligned}$$

This last line shows that the forced twist-and-flip has an integral invariant. If the function $b(t) = b_0$ is a constant then we have

$$\|\mathbf{u}(t)\| = \left\|A\left(\int_0^t \|\mathbf{u}(s)\|ds\right)\mathbf{u}_0\right\| = \|\mathbf{u}_0\|$$

and hence

$$\mathbf{u}(t) = A(\mathbf{u}_0\, t)(\mathbf{u}_0)$$

which is the twist part of the twist-and-flip map, when evaluated at $t = \pi/\omega$.

We now show that the function we have obtained in this implicit equation is the solution of the forced twist-and-flip ODE stated at be beginning of this section.

Recall that

$$\mathbf{z}_{k+1} = A(hr_k)(\mathbf{z}_k - \mathbf{s}(kh)) + \mathbf{s}(kh)$$

for small h. This is

$$\begin{aligned}\mathbf{z}_{k+1} - \mathbf{z}_k &= A(hr_k)(\mathbf{z}_k - \mathbf{s}(kh)) + \mathbf{s}(kh) - \mathbf{z}_k \\ &= A(hr_k)(\mathbf{z}_k - \mathbf{s}(kh)) - (\mathbf{z}_k - \mathbf{s}(kh)) \\ &= (A(hr_k) - I)(\mathbf{z}_k - \mathbf{s}(kh)),\end{aligned}$$

and hence

$$(\mathbf{z}(t+h) - \mathbf{z}(t))/h = (A(hr(t)) - I)(\mathbf{z}(t) - \mathbf{s}(t))/h.$$

Taking limits as $h \to 0$ we have

$$\dot{\mathbf{z}}(t) = \|\mathbf{z}(t) - \mathbf{s}(t)\| B(\mathbf{z}(t) - \mathbf{s}(t)),$$

where B is the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad \blacksquare$$