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ON SOLVING NONLINEAR FUNCTIONAL, FINITE DIFFERENCE, COMPOSITION, AND ITERATED EQUATIONS

RAY BROWN*
4650 North Washington Boulevard #606, Arlington, VA 22201 USA

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Abstract

In this letter, we present a general method for solving a wide range of nonlinear functional and finite difference equations, as well as iterated equations such as the Hénon and Mandelbrot equations. The method extends to differential equations using an Euler approximation to obtain a finite difference equation.

1. BACKGROUND

The first-order, autonomous, nonlinear finite difference equation

$$y_{k+1} = 2\,y_k^2 - 1, \quad y_0 \in [-1, 1] \tag{1}$$

is solved by

$$y_k = \cos(2^k \arccos(y_0))\,. \tag{2}$$

This is equivalent to the iterated, chaotic, dynamical system equation defined by the map[a]

$$T(x) = 2\,x^2 - 1, \quad x \in [-1, 1]\,.$$

The functional/duplication equation

$$f(2\,x) = 2\,f(x)^2 - 1, \quad f(0) = 1, \quad f'(0) = 0 \tag{3}$$

is solved by $f(x) = \cos(x)$. The iterated equation

$$g(g(x)) = 2\,x^2 - 1 \tag{4}$$

has as a solution

$$g(x) = \cos(\sqrt{2}\arccos(x)), \quad x \in [\cos(\pi/\sqrt{2}), 1]\,. \tag{5}$$

All of these solutions have $\cos(x)$ in common, a function which satisfies the duplication equation [Eq. (3)]. Functions which satisfy duplication equations are the basis for the solution of these other

*E-mail: raybrown@erols.com
[a]This observation is indirectly suggested by a note from Ulam and Von Neumann.[1]

three classes of problems, suggesting a general approach to their solution. In this letter, we set forth this approach.

2. THE GENERAL CASE

We begin by writing down the four classes of problems.

The general nonlinear, autonomous, finite difference equation and iterated dynamical system equation is given by:

$$y_{k+1} = G(y_k)\,. \tag{6}$$

The associated general functional equation is

$$f(a\,x) = G(f(x))\,. \tag{7}$$

The iterated equation is

$$g(g(g(\dots g(x)))) = G(x), k \text{ compositions}\,. \tag{8}$$

If we are able to find a locally invertible solution of Eq. (7), then Eq. (6) is solved by

$$y_k = f(a^k f^{-1}(y_0)) \tag{9}$$

and Eq. (8) is solved by

$$g(x) = f(a^{1/k} f^{-1}(x)) \tag{10}$$

when f maps the complex plane into the domain of f^{-1}. Particular solutions then result from establishing the appropriate interval on which $f^{-1}(x)$ exists.

3. POWER SERIES SOLUTIONS OF DUPLICATION EQUATIONS

The problem of solving Eq. (7) can be reduced to the question: "How large of a problem does G pose?" We make this problem tractable here by requiring that G be a polynomial. This covers a wide range of interesting cases.

If $f(a\,x) = G(f(x))$, and we assume that there exists a non-constant analytic solution, we may obtain the terms of the Taylor series by direct computation. The process of obtaining the power series will require that the value of a be fixed at some point, and that $f(0), f'(0)$ be fixed too. Given these values, the remaining derivatives at 0 follow from differentiating the duplication equation. For example, the possible values for $f(0)$ are obtained from the polynomial in $f(0)$:

$$f(0) = G(f(0))\,. \tag{11}$$

If λ_n are the roots of this equation, then $f(x) = \lambda_n$ are all solutions of the duplication equation. As we are seeking locally invertible solutions, these solutions are discarded, hence some derivative must be non-zero at 0. Given $f(0)$, we turn to the value of a and $f'(0)$ using

$$f'(a\,x)\,a = G'(f(x))f'(x)\,.$$

Setting $x = 0$, we have

$$f'(0)\,a = G'(f(0))f'(0)\,.$$

Possible solutions are $a = G'(f(0))$, and $f'(0) = 1$, or $f'(0) = 0$, thus deferring the determination of a until later. The choice $f'(0) = 1$ determines that the duplication constant a is $G'(f(0))$, which may be a complex number depending on the roots of $f(0) = G(f(0))$. If a is given in advance and $a \neq G(f(0))$, then $f'(0) = 0$. The second derivative equation is

$$f''(a\,x)\,a^2 = G''(f(x))f'(x)^2 + G'(f(x))f''(x)\,.$$

For $x = 0$, we have

$$f''(0)(a^2 - G'(f(0))) = G''(f(0))f'(0)\,.$$

Clearly, this process can be continued to obtain all Taylor coefficients. At each juncture, we have a choice of fixing a or fixing a derivative. Thus, there are numerous possible solutions depending on a. By fixing a in advance, we may only have the roots of Eq. (11) as solutions.

Once the formal Taylor series has been found, two problems remain: To find the radius of convergence of the series and the domain of a local inverse of f. These theoretical questions will not be discussed in detail here. The short answer is that there will generally be a positive radius of convergence, which may be the entire complex plane, and there will be a local inverse. We will illustrate these facts with some examples.

4. EXAMPLES

Example 1. Let $f(ax) = 2\,f(x)^2 - 1$, then $f(0) = 1, -1/2$. We choose $f(0) = 1$. Differentiating, we get $a\,f'(a\,x) = 2\,f(x)f'(x)$, and

$a\,f'(0) = 4\,f(0)f'(0)$. Choose $f'(0) = 1$, and then $a = 4$. Continuing, we get the power series for f:

$$f(x) = \sum_0^\infty b_k \frac{x^k}{k!}\,, \quad 1/b_k = \prod_{j=1}^{k}(2\,j-1)\,. \qquad (12)$$

The first few terms of this series are

$$1 + x + \frac{x^2}{2!\cdot 1\cdot 3} + \frac{x^3}{3!\cdot 1\cdot 3\cdot 5}\cdots.$$

Clearly, the series is uniformly convergent in the complex plane.

If we choose $f'(0) = 0$ and $f''(0) = -1$, then $a = 2$, and we obtain the cosine series. Non-invertible solutions are the constant functions $f(x) = 1$ and $f(x) = -1/2$, which also solve the associated nonlinear finite difference equation.

Example 2. Find a function g such that $g(g(x)) = x^2 + b$, where b is a constant.

First solve the duplication equation for f:

$$f(ax) = f(x)^2 + b\,.$$

We expect that a will depend on b. We first note that for generating the derivative duplication equations, we need only consider $b = 0$ as b enters into the formulae through $f(0)$. We write down the equations for five derivatives:

$$\begin{aligned}
f'(a\,x)\,a - 2\,f(x)f'(x) &= 0\\
f''(a\,x)\,a^2 - 2\,f(x)f''(x) &= 2\,f'(x)^2\\
f^{(3)}(a\,x)\,a^3 - 2\,f(x)f^{(3)}(x) &= 6\,f'(x)f''(x)\\
f^{(4)}(a\,x)\,a^4 - 2\,f(x)f^{(4)}(x) &\\
&= 8\,f'(x)f^{(3)}(x) + 6\,f''(x)^2\\
f^{(5)}(a\,x)\,a^5 - 2\,f(x)f^{(5)}(x) &\\
&= 10\,f'(x)f^{(4)}(x) + 20\,f''(x)f^{(3)}(x)\,.
\end{aligned}$$

We note that

$$f^{(n)}(a\,x)\,a^n - 2\,f(x)f^{(n)}(x) = \sum_{j=1}^{N} c_j\, f^{(N-j)}(x)f^{(j)}(x)$$

where $N = n-1$. The coefficients c_j can be obtained from a polynomial P_N, where

$$\mathrm{P}_N(x) = \sum_{j=1}^{N} c_j\, x^j$$

which is obtained by iteration of the functional shift mapping

$$\mathrm{S}(h(x)) = (x+1)h(x) + 2$$

where $h(x)$ is a function of x. In particular,

$$\mathrm{P}_N(x) = \mathrm{S}^{N-2}(2)\,.$$

For the special case $b = 0$, $f(x) = \exp(x)$ and $g(x) = f(\sqrt{2}f^{-1}(x)) = x^{\sqrt{2}}$.

Example 3. Find the solution to the mapping

$$x_{k+1} = \lambda\, x_k(1-x_k), \quad 1 < \lambda\,.$$

We seek a locally invertible analytic function satisfying the duplication equation

$$f(a\,x) = \lambda\, f(x)(1-f(x))\,.$$

The polynomial equation for $f(0)$ gives $f(0) = 0, 1-1/\lambda$, and we exclude the initial condition $f(0) = 0$, the constant solution.

$$f'(a\,x)\,a + (2\lambda\, f(x) - \lambda)f'(x) = 0$$

we may choose $f'(0) = 1$, then $2\lambda f(0) - \lambda = -a$.

$$f''(0) = -\frac{2\,\lambda}{a^2 - a}$$

and

$$f^{(3)}(0) = \frac{3(2\lambda)^2}{(a^3-a)(a^2-a)}\,.$$

5. GENERALIZATIONS

The equation

$$y_{k+1} = G(y_k)$$

is also solved by this method when G is not a polynomial but has all derivatives.

More importantly, the method extends to any number of dimensions. Let $X \in \mathbf{R}^n$ and $G : \mathbf{R}^n \to \mathbf{R}^n$ be a C^∞. Then the solution of the duplication equation

$$F(A\,X) = G(F(X)) \qquad (13)$$

where A is a matrix, provides the basis for solving the nth order nonlinear, autonomous, finite difference equation

$$X_{k+1} = G(X_k) \qquad (14)$$

and the composition equation

$$h^k(X) = G(X) \tag{15}$$

where h^k is h composed with itself k times. The solutions are

$$X_k = F(A^k F^{-1}(X_0))$$

and

$$h(X) = F(A^{1/k} F^{-1}(X))$$

respectively, when F maps n-dimensional space into the domain of F^{-1}. Note that the radius of convergence and domain of the inverse are to be determined. In many cases, the matrix A may be taken to be a diagonal matrix D. Extensions to general linear spaces would require that the matrix A be replaced by a bounded linear operator L on a Banach space.

Example 4. The Hénon map is given by

$$G\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 + y - c\,x^2 \\ d\,x \end{pmatrix}.$$

Iteration of this equation leads to the finite difference equation

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} 1 + y_k - c\,x_k^2 \\ d\,x_k \end{pmatrix}.$$

We form the duplication equation with a diagonal matrix:

$$F\begin{pmatrix} a\,x \\ b\,y \end{pmatrix} = \begin{pmatrix} f(a\,x, b\,y) \\ g(a\,x, b\,y) \end{pmatrix} = G(F(X)) \tag{16}$$

where

$$G(F(X)) = \begin{pmatrix} 1 + g(x,y) - c\,f(x,y)^2 \\ d\,f(x,y) \end{pmatrix}.$$

Extending the ideas of the first order case, we solve for $f(0,0), g(0,0)$ using the duplication equation. We proceed to solve for the Taylor coefficients of f, g by the same method as the first order case, using partial derivatives instead of ordinary derivatives. Doing this, we see that $g(0,0) = d\,f(0,0)$, and $c\,f(0,0)^2 - 2\,f(0,0) + 1 = 0$. As in the first order case, $f(0,0)$ may have multiple solutions. Next, we differentiate f, g partially with respect to x, y to get

$$f_x(a\,x, b\,y) = g_x(x,y) - 2\,c\,f(x,y) f_x(x,y)$$

leading to

$$\begin{aligned} a\,f_x(0,0) &= g_x(0,0) - 2\,c\,f(0,0) f_x(0,0) \\ b\,f_y(0,0) &= g_y(0,0) - 2\,c\,f(0,0) f_y(0,0) \\ a\,g_x(0,0) &= d\,f(0,0) \\ b\,g_y(0,0) &= d\,f(0,0)\,. \end{aligned}$$

Continuing, we derive all necessary partial derivatives. In the event that A must be chosen as a non-diagnonal matrix to assure the existence of a locally invertible solution, the computation of derivatives becomes more tedious, but poses no additional theoretical problems beyond the first order one-dimensional case.

Once this has been completed and we have obtained the map F, we have the result

$$H^k(X_0) = F(A^k F^{-1}(X_0))$$

where $H(X)$ is the Hénon map. Dropping subscripts, we get the factorization of the Hénon map:

$$H^k(X) = F(A^k F^{-1}(X))\,.$$

This equation states that H is conjugate to the map defined by A on the appropriate subspace of $\mathbf{R}^2$. The differentiable conjugacy is given by F. If the matrix A is hyperbolic, F maps two-dimensional space onto a bounded subset, and F^{-1} exists over a sufficient range, then we have proof that H is chaotic using only vector calculus.

6. THE CONTINUOUS CASE

We discuss this case in two parts. The first part will be the reduction of the continuous case to the discrete case. The second part will treat the continuous case directly with minimal results.

6.1 Reduction to the Discrete Case

We illustrate the reduction to the discrete case with the Rössler equation.

The Rössler equation is given by

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} -(y + z) \\ x + \alpha\,y \\ 2 + z\,(x - 4) \end{pmatrix}.$$

We construct the Euler approximation as

$$T(X) = X + \tau G(X)$$

where X is (x, y, z), τ is a small step size, and $G(X)$ is given by the right-hand-side of the preceding ODE.

We now seek a mapping F of $\mathbf{R}^3$ with the property

$$F(A\,X) = F(X) + \tau G(F(X))\,,$$

for some 3×3 matrix of constants. For $X = 0$, we have

$$F(0) = F(0) + \tau G(F(0))$$

so that $G(F(0)) = 0$. Thus, we must choose $F(0)$ as a fixed point of the mapping T. Differentiating, we get the equation

$$\mathbf{D}F(A\,X)\,A = \mathbf{D}F(X) + \tau \mathbf{D}G(F(X))\,\mathbf{D}F(X).$$

The choice $A = \mathbf{I} + \tau\,\mathbf{D}G(F(0))$ requires that $\mathbf{D}F(0)$ commute with $\mathbf{D}G(F(0))$, the derivative of the Rössler vector field at the chosen fixed point, $F(0)$. $\mathbf{D}F(0)$ may then be chosen to be invertible, thus assuring an inverse to F near 0. Continuing with higher derivatives we obtain a Taylor series for F and conclude that

$$A = \begin{pmatrix} 1-\tau & -\tau & \\ \tau & 1+\alpha\tau & 0 \\ z_1 & 0 & 1+(x_1-4)\tau \end{pmatrix}$$

and

$$\mathbf{D}F(0) = \begin{pmatrix} c_1 & c_2 & c_3 \\ g_1 & g_2 & g_3 \\ h_1 & h_2 & h_3 \end{pmatrix}$$

where the c_i are arbitrary constants and the g_i, h_i depend on the c_i.

Collecting these results, we see that we may take the step size τ, as small as we like to get a good approximation to the continuous dynamics. Then, the solution of the Rössler equation at the kth step is given by

$$T^k(X_0) = F(A^k F^{-1}(X_0))$$

if the appropriate series converges and inverse exists. For $\alpha > 0$, A is hyperbolic. But, a complete proof of chaos requires us to prove that the Taylor series converges, that F^{-1} exists over a sufficiently large region, and that F maps all of three space into the domain of F^{-1}. This will depend on the value of α, and the values of the c_i. These theoretical matters are not investigated in this letter.

6.2 The Continuous Case

Consider the equation

$$x(x_0, t) = \cos(2^t \arccos(x_0))\,.$$

Formally differentiating, we have

$$\dot{x} = -\sin(2^t \arccos(x_0))2^t\, \ln(2) \arccos(x_0)\,.$$

For all t, x_0, where the cosine is invertible, this reduces to

$$\dot{x} = -\sin(\arccos(x)) \ln(2) \arccos(x)$$

which is an autonomous first order equation. The "solution" is chaotic but not unique. In general, functions of the form

$$x(x_0, t) = f(\exp(At) f^{-1}(x_0)) \tag{17}$$

where x is n-dimensional, and A is an $n \times n$ matrix, define one-parameter groups, in that

$$x(x(x_0, s), t) = x(x_0, s+t)$$

so long as f is invertible. Solutions of automonous differential equations always define such one-parameter groups. Formally differentiating, Eq. (17) gives rise to the autonomous ODE

$$\dot{x} = f'(f^{-1}(x))\,A\,f^{-1}(x)\,.$$

If f is globally invertible, the solution is linear after a change of coordinates. The interesting cases arise when f is locally, but not globally invertible, and f maps all of n-dimensional space onto a bounded subset of the domain of f^{-1}. The various solutions of an ODE are then determined by the separate functions that can be derived from a duplication equation. In the case of the equation $x_{k+1} = 2\,x_k^2 - 1$, there are many "solutions," depending on the initial conditions, and this fact is reflected in the different solutions to the associated duplication equation.

7. CONCLUSIONS

We conclude from these results that all four problems are solvable in a broad array of cases in any number of dimensions. In particular, we have presented a method of solving the n-dimensional, general first order, autonomous, nonlinear finite difference equation,

$$X_{k+1} = G(X_k)\,.$$

First, the associated duplication equation is formed. This equation will have multiple solutions. Example 1 is representative of the types of solutions that are possible. Select the solution F, such that F maps all of n-dimensional space into the domain of F^{-1}. For Example 1, this is the cosine solution. This F will then provide the solution as in Eqs. (9) and (10).

Viewed in another light, the method presented for solving iterated equations is equivalent to assuming that the solution is C^∞ and locally analytically conjugate to an iterated matrix of complex constants, and then solving for the conjugating map as a Taylor series. There are a wide array of cases where this approach will be valuable, and also much simpler than attempting to prove the existence of horseshoes. Of particular importance is that the method relies computationally on nothing more than vector calculus.

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