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GENERALIZATIONS OF THE CHUA EQUATIONS

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In this paper we present two generalizations of the equations governing Chua's circuit. In order to obtain the first generalization we simplify Chua's equations by replacing the piecewise-linear term with a signum function. The resulting simplified system produces a double scroll similar to the one observed experimentally in Chua's circuit. What is significant about this simplified system is that it can be reduced to what we shall call a *two-dimensional single scroll*, and from the two-dimensional single scroll we are able to derive a one-dimensional map. This entire derivation is carried out *analytically*, in contrast to the one-dimensional map analysis that has been carried out for the Lorenz equations which is based on *axioms*.

After we have carried out our analysis for this simplified version of Chua's equations, we use these equations as a guide to the construction of the first generalization to be presented in this paper. We call this a type-I generalization of Chua's equations. The generalization consists in using a two-dimensional autonomous flow as a component in a three-dimensional autonomous flow in such a way that the resulting equations will have double scroll attractors similar to those observed experimentally in Chua's circuit.

The value of this generalization is that: (1) it provides a *building block* approach to the construction of chaotic circuits from simpler two-dimensional components which are not chaotic by themselves. In so doing it provides an insight into how chaotic systems can be built up from simple nonchaotic parts; (2) it illustrates a precise relationship between three-dimensional flows and one-dimensional maps. Of particular significance in this regard is a recent paper of Misiurewicz [1993], which analytically connects the two-dimensional single scroll to the class of unimodal maps, thus providing a framework within which a theory linking unimodal maps to three-dimensional flows may be possible.

The second generalization is suggested by considering three-dimensional flows whose only nonlinearities are sigmoid, sgn, or piecewise-linear functions. Clearly, such flows are a generalization of the Chua equations. We call these equations type-II generalization Chua equations. The significance of this direction of investigation is that attractors similar to the Lorenz and Rössler attractors can be produced from type-II generalized Chua equations in a building block approach using only piecewise-linear vector fields. As a result we have a method of producing the Lorenz and Rössler dynamics in a circuit without the use of multipliers. This suggests that the type-II generalized Chua equations are in some sense fundamental in that the dynamics of the three most important autonomous three-dimensional differential equations producing chaos are seen as variations of a single class of equations whose nonlinearities are generalizations of the Chua diode.

1. Introduction

In this paper we explore two generalizations of Chua's equations. These two generalizations are significant in that, (1) they show how chaotic flows in three dimensions can be constructed from simple nonchaotic parts; (2) how three-dimensional flows can be analytically related to one-dimensional maps; and (3) how a very wide range of chaotic dynamical systems, including systems similar to those at Rössler and Lorenz, can be constructed from piecewise-linear flows. This is significant for many reasons: prior to these results, the number of chaotic dynamical systems available to guide research into the mechanisms of chaos were very few. Second, in order to construct new systems with given properties there was no available methodology that started with simple components and, in a systematic way, used these components to build new chaotic systems. Third, there was no analytical connection between three-dimensional flows and one-dimensional maps that offered the prospect of linking the relatively extensive machinery of one-dimensional map theory to three-dimensional flows. The results of this paper provide an advance in all three of these research areas.

As noted by Michael Hirsch of Princeton, the flows described here "almost" have solutions. At least these flows come as close as possible to having constructable solutions as a chaotic differential equation can be expected to come.

It is historically noteworthy that the Chua circuit has provided the basis for the research presented here and that had this circuit not existed and we were to have only the equations of Rössler and Lorenz to guide us, these simple constructions might never have been observed.

The dimensionless form of Chua's equations are given by

$$\begin{aligned} \dot{x} &= \alpha(y - x - f(x))\,, \\ \dot{y} &= x - y + z, \\ \dot{z} &= -\beta y, \end{aligned} \tag{1}$$

where,

$$f(x) = \begin{cases} bx + a - b & \text{for} \quad x \geq 1.0, \\ ax & \text{for} \quad |x| \leq 1.0, \\ bx - a + b & \text{for} \quad x \leq 1.0, \end{cases} \tag{2}$$

is a three segment piecewise-linear function and α, β are dimensionless parameters.

We note from the history of Chua's circuit [Chua, 1993] that the function $f(x)$ represents a *nonlinear* resistor, and hence can be *any scalar* function of one variable. The choice of the piecewise-linear function (2) is only for convenience in synthesizing the physical circuit. It is also obvious from Chua [1993] that $f(x)$ can be modified in many ways without changing the qualitative dynamics. In Sec. 2 we will choose the discontinuous "signum" function $\mathrm{sgn}(x)$ for $f(x)$.

Section 2 of this paper discusses the type-I generalized Chua equation.

In Sec. 2.1 we derive a simplified version of Chua's circuit equations. In Sec. 2.2 we derive what we call a three-dimensional single scroll and a two-dimensional single scroll. In Sec. 2.3 we show how to reduce the two-dimensional single scroll to a one-dimensional map. In Sec. 2.4 we extend the definition of the type-I generalized Chua equations based on Misiurewicz [1993]. In Sec. 2.5 we provide a more formal definition of the two-dimensional single scroll. In Sec. 2.6 we prove a Lemma about the parametric representation of the one-dimensional maps obtained from the two-dimensional single scrolls. In Sec. 2.7 we demonstrate how to derive more general double scrolls from two-dimensional single scrolls based on the equations of Duffing and Van der Pol. We also provide a formal definition of type-I generalized Chua equations.

In Sec. 3 we define the type-II generalized Chua equations and in Sec. 3.1 we derive a type-II generalized Chua equation which has an attractor very similar to that of Rössler. In Sec. 3.2 we derive type-II generalized Chua equations with an attractor similar to that of the Lorenz system.

In the appendix we show how to derive simple maps that are useful in studying the dynamics of type-I and type-II generalized Chua equations. We call them generalized Chua maps.

2. Type-I Generalization of Chua's Equations

In this section we derive the type-I generalized Chua equations, the single scroll, and relate them to one-dimensional maps. We show how to derive a double scroll from a large class of two-dimensional autonomous flows.

2.1. *Simplification of Chua's Equation*

The dimensionless Chua equations can be recast into the form

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{bmatrix} -\alpha & \alpha & 0.0 \\ 1.0 & -1.0 & 1.0 \\ 0.0 & -\beta & 0.0 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \alpha \begin{bmatrix} f(x) \\ 0.0 \\ 0.0 \end{bmatrix}. \tag{3}$$

A more compact expression for $f(x)$ is given by $bx + 0.5(a-b)(|x+1.0| - |x-1.0|)$. Using this expression Eq. (3) simplifies to

$$\dot{x} = \begin{cases} \mathbf{A}(\alpha, \beta, b)(x-k) & \text{for} \quad x \geq 1, \\ \mathbf{A}(\alpha, \beta, a)x & \text{for} \quad |x| \leq 1, \\ \mathbf{A}(\alpha, \beta, b)(x+k) & \text{for} \quad x \leq -1, \end{cases} \tag{4}$$

where

$$\mathbf{A}(\alpha, \beta, c) = \begin{bmatrix} -\alpha(c+1) & \alpha & 0.0 \\ 1.0 & -1.0 & 1.0 \\ 0.0 & -\beta & 0.0 \end{bmatrix},$$

$$\mathbf{k} = \begin{bmatrix} k \\ 0 \\ k \end{bmatrix}, \tag{5}$$

$c = b$ or a depending on the value of x, and $k = (b-a)/(b+1)$. Thus the vector field determined by the matrix $\mathbf{A}$ varies depending on which of three regions the vector $\mathbf{x}$ is in. Figure 1(a) is the well-known attractor for this equation.

We seek to simplify Eq. (4) to a set of equations having only two linear regions, instead of three, and such that the matrix $\mathbf{A}$ is the same for both regions. In doing this we want to preserve the qualitative dynamics found in the Chua circuit.

In particular we want to introduce a function in place of $f(x)$ for which the region where $c = a$ is such that $|x| \leq \epsilon$, where ϵ is a number we can control. If we can do this and then decrease ϵ to zero, while holding on to the global dynamics found in the Chua circuit we will have a simpler equation with which to work. In doing this it can happen that the system can go unbounded for reasons that will be clear later. So that if we decrease the middle region, we must also decrease the real part of the expanding eigenvalues (which are complex conjugate) in order to retain a region where there are bounded solutions of the ODE. We can carry out this strategy if we replace the piecewise-linear function

$$h(x) = 0.5(|x+1.0| - |x-1.0|) \tag{6}$$

that makes up part of $f(x)$ by the C^∞ function

$$g(x) = \frac{\exp(\gamma x) - 1.0}{\exp(\gamma x) + 1.0}. \tag{7}$$

(This function is known as a *sigmoid* function due to its "S" shaped graph and it is prevalent in the theory of neural networks.)

Doing this we have the following equation:

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{bmatrix} -\alpha(b+1) & \alpha & 0.0 \\ 1.0 & -1.0 & 1.0 \\ 0.0 & -\beta & 0.0 \end{bmatrix} \times \begin{pmatrix} x - kg(x) \\ y \\ z + kg(x) \end{pmatrix}. \tag{8}$$

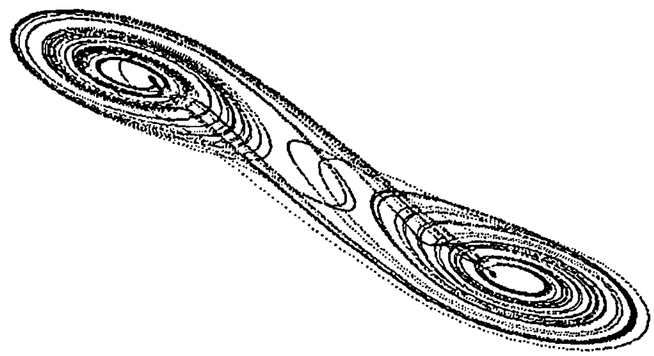

Fig. 1(a). The double scroll attractor.

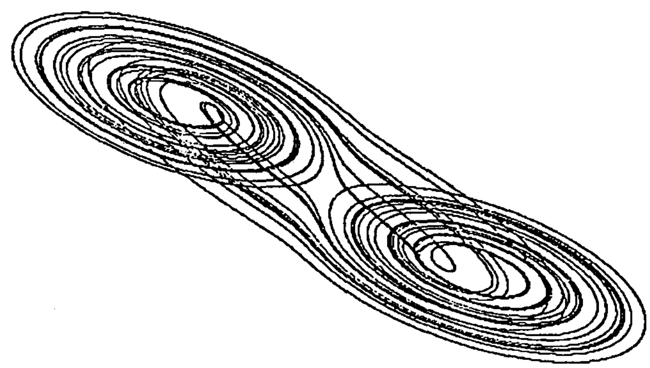

Fig. 1(b). The sigmoid double scroll, $\gamma = 10.0$, $\beta = 15.0$.

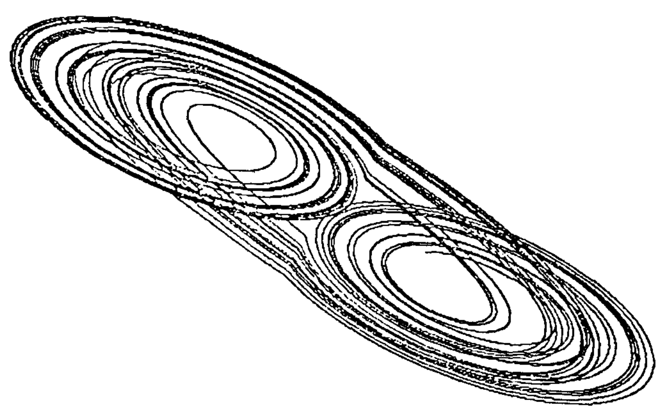

Fig. 1(c). The sigmoid double scroll, $\gamma = 100.0$, $\beta = 15.0$.

By replacing the piecewise-linear function Eq. (2) with a sigmoid function Eq. (7) and slightly increasing β, we can obtain an equation which will produce chaotic dynamics very similar to those in the Chua equations, i.e., produces a double scroll.

After this replacement is made we study what happens as γ is increased. We begin by setting $\gamma = 10.0$, which compresses the middle region, and to offset this we increase β from 14.2857 to 15.0 in order to lower the rate of expansion. Figure 1(b) shows the attractor for this system. Having made our replacement successfully we increase γ to 100.0 (β will need no further adjustments) and obtain Fig. 1(c). This equation has bounded solutions which are analytic, and for which the middle region, i.e., the region for which $|g(x)| < 1$, is very small, and, in fact, its size is inversely proportional to γ. It is a simple matter now to let $\gamma \to \infty$ to obtain the following simplification of Chua's original equation, Eq. (3):

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{bmatrix} -\alpha(b+1) & \alpha & 0.0 \\ 1.0 & -1.0 & 1.0 \\ 0.0 & -\beta & 0.0 \end{bmatrix} \times \begin{pmatrix} x - k\,\mathrm{sgn}(x) \\ y \\ z + k\,\mathrm{sgn}(x) \end{pmatrix} . \tag{9}$$

Equation (9) is a pointwise limit, rather than the uniform limit of Eq. (8) but that is not a limitation on its use since Eq. (9) satisfies all of our requirements, most important of which is that it has a double scroll attractor just as in Chua's original equations. Even though when $\gamma = \infty$ the system has a discontinuity at $x = 0.0$, this discontinuity does not introduce chaos where it did not already exist: If we fix a specific time T in the future, there exists a γ defining a C^∞ equation having a solution arbitrarily close to the solutions of Eq. (9) for all time less than T. As a result of these numerical observations, we have every right to believe that Eq. (9) is a system which is representative of the chaotic dynamics of Chua equation. As Fig. 2(a) shows, the simplified system produces a double scroll.

All constants in Eq. (9) are the same as those in the original Chua equations, Eq. (3), with the exception that $\beta = 15.0$ rather than 14.2857. In particular, $a = -8/7$, $b = -5/7$, $\alpha = 9.0$, $-\alpha(b+1) = -18/7$, and $k = 1.5$. For comparison, we recast Chua's original equations, Eq. (1), in the same form as Eq. (9) in order that their differences be easily discerned:

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{bmatrix} -\alpha(b+1) & \alpha & 0.0 \\ 1.0 & -1.0 & 1.0 \\ 0.0 & -\beta & 0.0 \end{bmatrix} \times \begin{pmatrix} x - kh(x) \\ y \\ z + kh(x) \end{pmatrix} , \tag{10}$$

where $h(x)$ is given by Eq. (6).

Figure 2(b) is the double scroll from Eq. (9) which has been rotated to better illustrate the effect of suppressing the middle linear region present in the Chua equation. Doing this makes the vector field discontinuous, but as we point out later, this does not change the complexity in the simplified version of the Chua equations from that in the

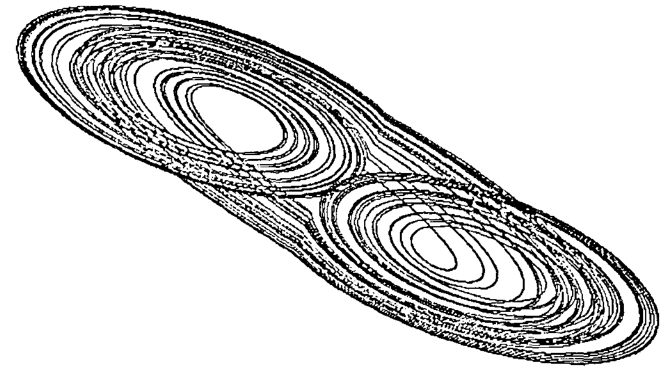

Fig. 2(a). The SGN double scroll, $\beta = 15.0$.

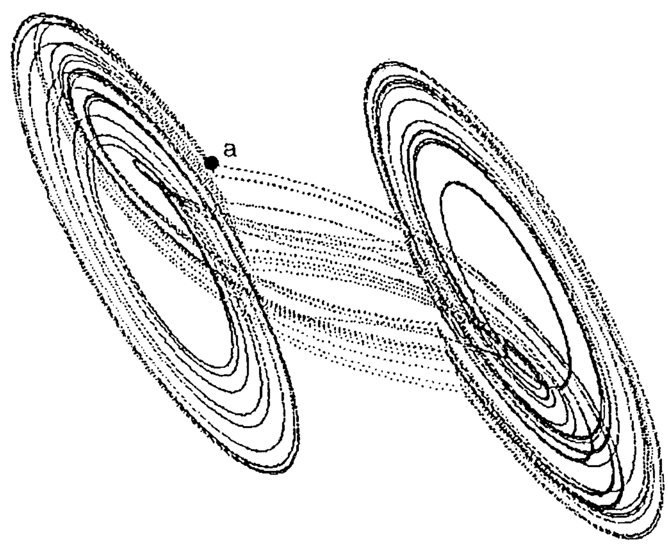

Fig. 2(b). The SGN double scroll rotated to reveal missing middle region, $\beta = 15.0$.

original Chua equations. Figure 2(b) illustrates where this discontinuity occurs and also shows that the effect of removing the middle region from the Chua equations is to make the transition between the two remaining regions of the scroll more distinct. What is clear is that we do have an object very similar to the double scroll, and by numerical examination, we are convinced that we still have a chaotic attractor.

The analytical advantage of working with Eq. (9) is that the matrix in Eq. (9) is constant while the function $\text{sgn}(u)$ takes on only two values. In contrast, in Eq. (10), while the matrix **A** has been made constant the function $h(u)$ is continuous and preserves the middle region.

For the matrix **A** in Eq. (9) we can introduce coordinate transformations that separate, or "decouples" from a circuit theoretic point-of-view, its invariant subspaces. These subspaces determine the stable and unstable manifolds for the linear vector field near the critical points $(-k, 0.0, k)$ and $(k, 0.0, -k)$. These two points were also critical points for the Chua equations. The dynamics of the Eq. (10) and Eq. (9) are the same at these points.

We know from linear algebra that we may write the matrix **A** in Eq. (9) as $\mathbf{JDJ}^{-1}$, where

$$\mathbf{J} = \begin{bmatrix} 1.0 & -1.287 & 1.0 \\ 0.143 & -1.513 & -0.148 \\ -2.37 & -2.14 & -0.569 \end{bmatrix}$$

and

$$\mathbf{D} = \begin{bmatrix} 0.0 & -9.876 & 0.0 \\ 1.0 & -0.334 & 0.0 \\ 0.0 & 0.0 & -3.9055 \end{bmatrix}.$$

Doing this and changing coordinates, Eq. (9) is transformed to

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{bmatrix} 0.0 & -9.876 & 0.0 \\ 1.0 & -0.334 & 0.0 \\ 0.0 & 0.0 & -3.9055 \end{bmatrix} \times \begin{pmatrix} x - 0.297\,k\,\text{sgn}(u) \\ y + 0.0363\,k\,\text{sgn}(u) \\ z - 0.656\,k\,\text{sgn}(u) \end{pmatrix}, \qquad (11)$$

where $u = x - 1.287y + z$. It should be noted that if we transform Eq. (10) by the same coordinates as we used to transform Eq. (9), the matrix **A** would be diagonalized but the function $h(u)$ would not be improved, and so the troublesome middle region would not be eliminated.

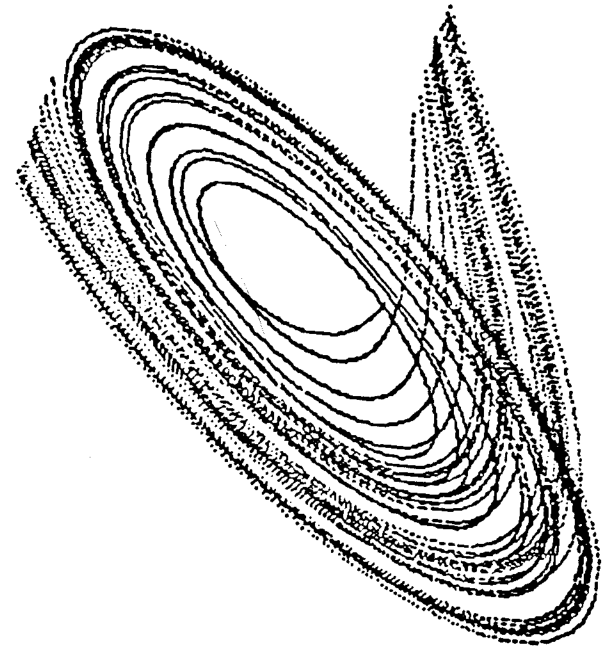

Fig. 3(a). The transformed SGN single scroll, $\beta = 15.0$, contracting eigenvalue $= -3.9055$.

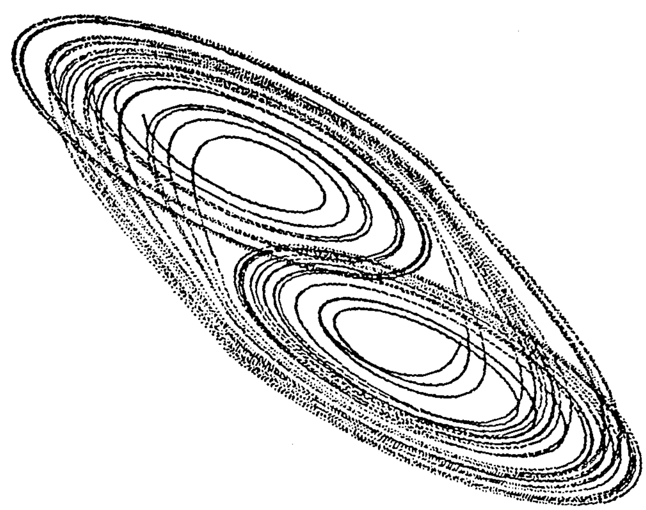

Fig. 3(b). The transformed SGN double scroll, $\beta = 15.0$.

Figure 3(a) is the double scroll for the transformed equations Eq. (11). No attempt has been made to rotate it to the same angle as Fig. 2(b), but it is clear that we still have a double scroll.

2.2. *Three-dimensional and two-dimensional single scrolls*

We now use the fact that the vector field defined by Eqs. (10) and (11) is an odd symmetric vector field and is invariant under the flip map $\mathbf{x} \to -\mathbf{x}$. This allows us to view the double scroll in Eq. (11) as a single scroll. In particular, whenever $\mathrm{sgn}(u)$ changes sign we apply the flip map and use the linear ODE:

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{bmatrix} 0.0 & -9.876 & 0.0 \\ 1.0 & 0.334 & 0.0 \\ 0.0 & 0.0 & -3.9055 \end{bmatrix} \times \begin{pmatrix} x - 0.297\ k \\ y + 0.0363\ k \\ z - 0.656\ k \end{pmatrix}, \quad (12)$$

to continue the orbit.

If as in the terminology of Brown & Chua [1991] we identify the last differential equation as an expanding linear twist, we see that we have *factored* Eq. (11) into the form FT, where the map T is determined by selecting an initial condition $\mathbf{x}_0$, integrating the above linear ODE until the solution starting at this initial condition reaches the boundary determined by the function $u = 0$ and using this *final* condition as the value of $\mathrm{T}(\mathbf{x}_0)$. We continue the solution by applying the flip map, F, to this final condition and using this flipped value as the initial condition for the above ODE. The effect of doing this is exactly the same as the process used in the twist-and-flip maps of Brown & Chua [1991]. In this way we are able to plot the entire double scroll of Eq. (11) on one side of the plane determined by the function $u = x - 1.287y + z = 0.0$ around only one of the fixed points, just as happens in using the twist-and-flip map.

Doing this amounts, mathematically speaking, to folding the three-dimensional space in half along the plane $x - 1.287y + z = 0.0$ and identifying the points $\mathbf{x}$ and $-\mathbf{x}$. That is, we consider the two points $\mathbf{x}$ and $-\mathbf{x}$ as the same point. Figure 3(b) shows a single trajectory of Eq. (12) for which this is done. We call this the *single* scroll.

Since we have separated out the stable manifold direction in the transformed equations we can examine the effect of increasing the contracting eigenvalue by considering the following equation instead of Eq. (11):

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{bmatrix} 0.0 & -9.876 & 0.0 \\ 1.0 & -0.334 & 0.0 \\ 0.0 & 0.0 & -\gamma \end{bmatrix} \times \begin{pmatrix} x - 0.297\ k\ \mathrm{sgn}(u) \\ y + 0.0363\ k\ \mathrm{sgn}(u) \\ z - 0.656\ k\ \mathrm{sgn}(u) \end{pmatrix}. \quad (13)$$

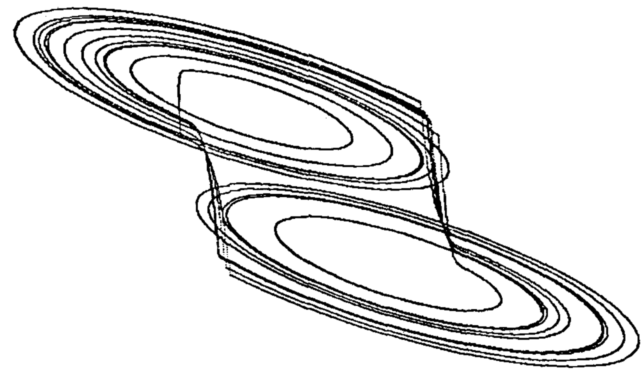

Fig. 4(a). The transformed SGN double scroll, $\beta = 15.0$, eigenvalue $= -100.0$.

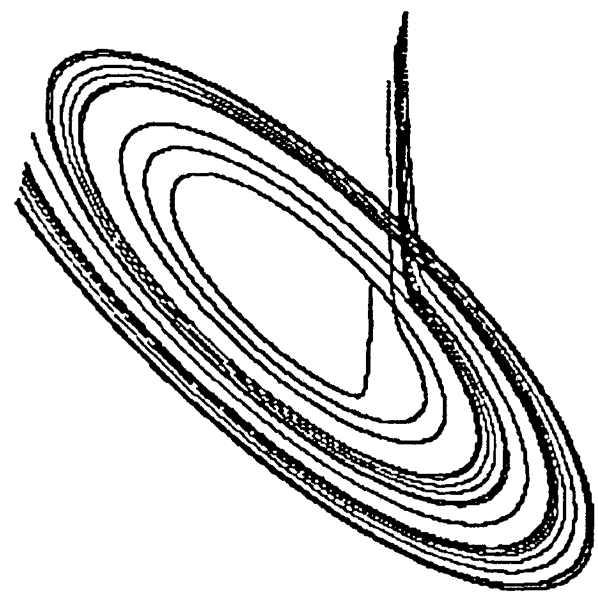

Fig. 4(b). The transformed SGN single scroll, $\beta = 15.0$, eigenvalue $= -1.00.0$.

In this equation the entry 3.9055 in Eq. (11) has been replaced by γ. (We are reusing the symbol "γ" in a manner that is unrelated to the "γ" in Eq. (7).)

Figure 4(a) shows the double scroll obtained from Eq. (13) with the eigenvalue $\gamma = 100.0$. The effect of increasing γ is a flatten the scroll onto a pair of parallel planes. If we combine this with the folding operation used in Fig. 3(b), we get the single scroll seen in Fig. 4(b) obtained from using Eq. (13) with $u > 0$ combined with the flip as was done with Eq. (12).

The point of this analysis is to conclude that the source of chaos in Eq. (11) and (13) can be understood by analyzing a two-dimensional single scroll which we obtain by considering the limit of

(13) as the contracting eigenvalue, $\gamma \rightarrow \infty$. As this happens we see that $z \rightarrow 0.984$ and we get a limiting two-dimensional single scroll on which all complex dynamics occur. The linear part of the two-dimensional single scroll is given by

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{bmatrix} 0.0 & -9.876 \\ 1.0 & 0.334 \end{bmatrix} \begin{pmatrix} x - 0.4455 \\ y + 0.054 \end{pmatrix}. \quad (14)$$

The nonlinear part is supplied by the condition that we apply the flip map when $\mathrm{sgn}(x - 1.287y + 0.984) < 0.0$.

Equation (14) is solved by

$$\begin{aligned} x(t) &= \exp(\alpha t/2)[(x_0 - 0.4455)\cos(\omega t) + C_1 \sin(\omega t)] \\ &\quad + 0.4455, \\ y(t) &= \exp(\alpha t/2)[(y_0 + 0.054)\cos(\omega t) + C_2 \sin(\omega t)] \\ &\quad - 0.054, \qquad (15) \end{aligned}$$

where

$$\begin{aligned} C_1 &= -(0.5\alpha(x_0 - 0.4455) + \beta(y_0 + 0.054))/\omega, \\ C_2 &= -((x_0 - 0.4455) - \frac{\alpha}{2\beta}0.5(y_0 - 0.054))/\omega, \end{aligned}$$

and $\alpha = 0.334$, $\beta = 9.876$, $\omega = \sqrt{\beta - (0.5\alpha)^2}$.

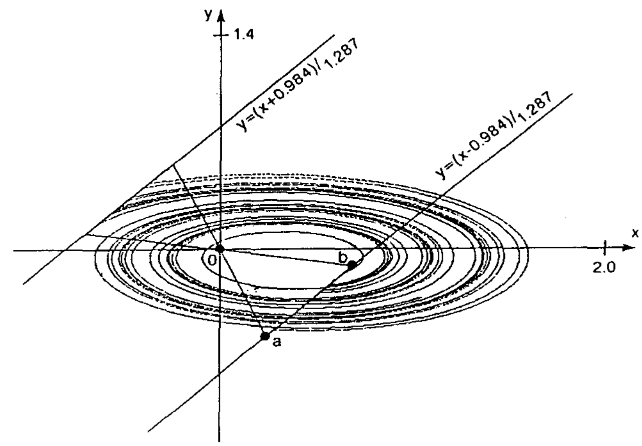

Fig. 5(a). The two dimensional SGN single scroll.

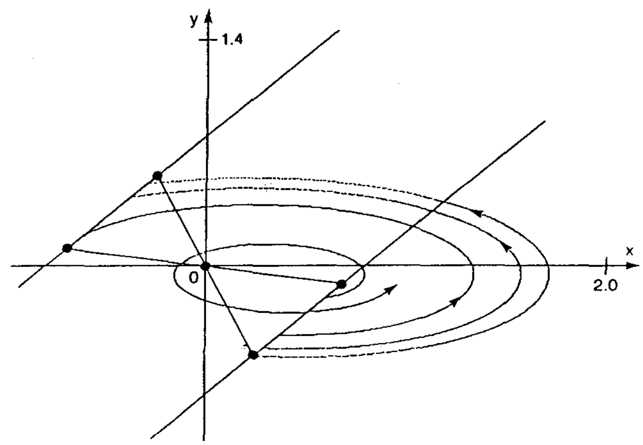

Fig. 5(b). The two dimensional SGN single scroll, four iterations of one orbit.

2.3. *One-dimensional maps from two-dimensional single scrolls*

Figure 5(a) Shows a single trajectory of the two-dimensional single scroll determined by Eq. (15). It also shows that the single scroll maps the line $y = (x - 0.984)/1.287$ to its image under the flip map. Figure 5(b) shows that for initial conditions of the form $0.3 \leq x \leq 0.85$ and $y = (x - 0.984)/1.287$, a segment of this line is mapped into itself. There are two fixed points on this line segment: $(0.54, -0.347)$ and $(0.6876, -0.2293)$. Figure 6(a) shows the first mentioned fixed point of this map and Fig. 6(b) shows the second fixed point. We have now associated Eq. (9) with a one-dimensional map of a segment of the line $y = (x - 0.984)/1.287$ onto itself.

We review how this one-dimensional map works: we begin with an initial condition on the line $y = (x - 0.984)/1.287$ with the value of the x coordinate in the closed interval $[0.3, 0.85]$. We use Eq. (15) to produce a trajectory which expands outward until it meets the line $y = (x + 0.984)/1.287$. We then apply the flip map, which takes this point on the line $y = (x + 0.984)/1.287$ back to the line $y = (x - 0.984)/1.287$, where the x value will lie in the closed interval $[0.3, 0.85]$. This flipped point will then be used as the initial conditions for Eq. (15) to generate a new trajectory. Hence we see that this line segment is mapped onto itself. From this we conclude that the source of the complexity, or chaos, in Eqs. (9), (11), and (13) can be traced to a one-dimensional map. Figure 6(c) shows a computer-generated graph of this one-dimensional map on the interval $[0.363, 0.85]$. The end point of the interval of the map has been moved forward since the

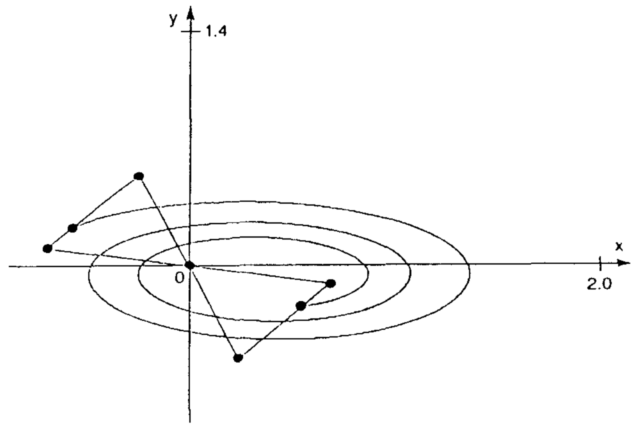

Fig. 6(a). The two-dimensional SGN single scroll, fixed point at (0.54, −0.347).

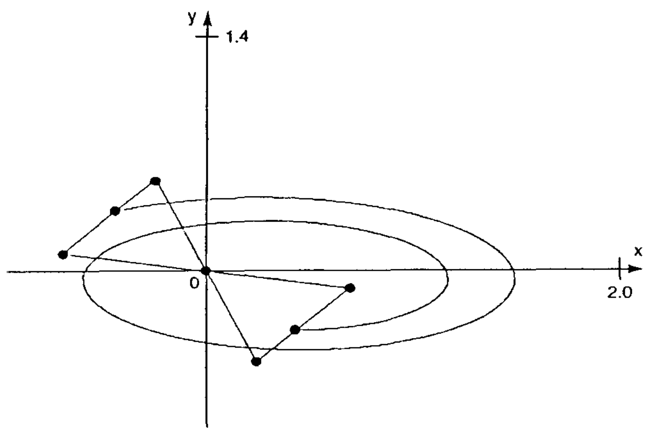

Fig. 6(b). The two-dimensional SGN single scroll, fixed point at (0.6876, −0.2293).

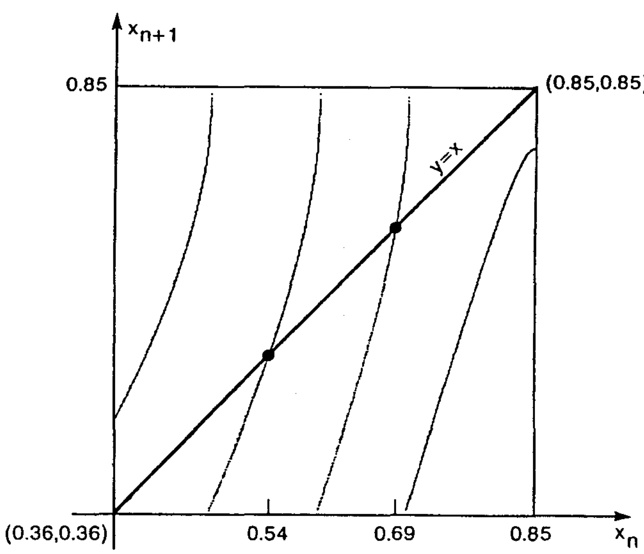

Fig. 6(c). Computer-generated graph of the one-dimensional map derived from the two-dimensional single scroll.

smaller interval [0.363, 0.85] is also mapped into itself. The one-dimensional map shows the existence of the two fixed points noted above. We will show in Sec. 2.6 that the graph of this map can be written down in a closed form parametric representation.

We may convert Eq. (13) into a smooth equation by replacing the "sgn" function by the sigmoid function, Eq. (7), thus obtaining a C^∞ vector field whose chaotic properties are closely tied to a given one-dimensional map so long as γ is large. The possibility that Eq. (13) could be reduced to a simple one-dimensional map was suggested by Prof. Morris Hirsch.

2.4. *Generalizing Chua's equation to include Misiurewicz's form of the two-dimensional single scroll*

The foregoing analysis and the work of Misiurewicz [1993] provides the motivation for introducing the definition of the type-I generalized Chua equations to designate an important class of three-dimensional autonomous ordinary differential equations very similar in their dynamics to the Chua equations. The key similarity is that this class of ODE's have attractors that look very similar to the double scroll. However, in contrast to the original Chua equations, the nonlinearity in these equations is composed of two linear components, instead of three, as found in Chua's equations.

We will define type-I generalized Chua equations in two steps. The first step is to include the following class of double scroll producing equations (motivated by Misiurewicz [1993]) within the definition of type-I generalized Chua equations:

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{bmatrix} s & -1.0 & 0.0 \\ 1.0 & s & 0.0 \\ 0.0 & 0.0 & -\gamma \end{bmatrix} \begin{pmatrix} x - a\,\mathrm{sgn}(u) \\ y - b\,\mathrm{sgn}(u) \\ z - \mathrm{sgn}(u) \end{pmatrix}, \tag{16}$$

where $u = z - x$, and a, b are any real constants, and s is a positive constant.

This form of the type-I generalized Chua equations based on the analysis of Misiurewicz [1993] reveals the role of the two-dimensional flow that defines the local unstable manifold located at the fixed point $(a, b, 1)$. This two-dimensional flow is given by the equation

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{bmatrix} s & -1.0 \\ 1.0 & s \end{bmatrix} \begin{pmatrix} x-a \\ y-b \end{pmatrix}, \tag{17}$$

which defines a source which spirals outward from the critical point (a, b). If, as in Misiurewicz [1993], we use Eq. (17) as the two-dimensional single scroll in the previous section and, in place of the line $y = (x - 0.984)/1.287$, we use the line $x = 1$ as the line at which we apply the flip map, then we obtain a mapping of the line $x = -1$ onto itself and the entire analysis of Misiurewicz is available for analyzing these equations. In the following section we present a Lemma about parametric equations for the graph of one-dimensional maps from this single scroll that derives its interest from the analysis in Misiurewicz [1993].

2.5. *Working definition of the two-dimensional single scroll*

The work of Misiurewicz suggests the need for a somewhat more formal definition of single scrolls. We note that Misiurewicz [1993] implicitly provides the rigorous mathematical definition for the single scroll, but this definition would require some translation for the nonspecialist and so we do not repeat it here.

Let the following differential equation define a vector field on $\mathbf{R}^2$:

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \mathrm{F}\begin{pmatrix} x \\ y \end{pmatrix}, \tag{18}$$

and assume that for any initial condition (x_0, y_0) in $\mathbf{R}^2$ this equation has a unique solution defined for all time $-\infty < t < \infty$. Also suppose that for each initial condition of the form $(-1.0, y_0)$ there is a time t at which the solution having this initial condition crosses the vertical line $x = 1$.

Working Definition. Two-dimensional Single Scroll. A two-dimensional single scroll is defined by the curve formed by starting with an initial condition of the form $(-1, y_0)$ and following the solution of Eq. (18) above until it meets the line $x = 1$, and then applying the flip map, and continuing in this manner indefinitely.

When we refer to a two-dimensional single scroll we mean that we are talking about an ODE with the above characteristics with which we generate an orbit using this ODE and the flip map as described in the above definition.

2.6. *Parametric equations for one-dimensional maps*

We now provide a Lemma for the parametric equations of one-dimensional maps that arise from two-dimensional single scrolls.

Lemma 1. *Assume that the following equation defines a single scroll*

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \mathrm{F}\begin{pmatrix} x \\ y \end{pmatrix}$$

and suppose that the solution of this equation (considered as an ODE and not a single scroll) is given by

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} f(x_0, y_0, t) \\ g(x_0, y_0, t) \end{pmatrix}.$$

Let $H(y_0, t) = f(-1, y_0, t) - 1$ *and* t' *be such that* $f(-1, y_0, t') = -1$, *and assume that*

$$\frac{\partial H}{\partial y}(y_0, t') \neq 0$$

and that this partial derivative at (y_0, t') *is continuous.*

Then the graph of the one-dimensional map defined by this single scroll has a local parametric representation of the form $(h(t), -y(t))$, *where*

$$\begin{pmatrix} y_0(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} h(t) \\ g(-1, y_o, t) \end{pmatrix}$$

for $t \in (t' - \delta, t' + \delta)$, *for some* $\delta > 0$.

Proof. By the implicit function theorem and the assumptions of our Lemma the implicit function $H(y_0, t) = 0$ can be solved as $y_0 = h(t)$ in some neighborhood of t. Hence, in this same neighborhood $y(t) = g(-1, h(t), t)$. The graph of the correct one-dimensional map is given by the ordered pairs $(h(t), -y(t))$. ■

Note that when the single scroll is defined by a linear differential equation, we are always able to obtain a closed form parametric representation of the graph of the one-dimensional maps since in the equation $f(-1, y_0, t) = 1$, y_0 only appears as a factor multiplying a function of t.

We may now ask what happens when we specify a one-dimensional map in advance and ask if there is a single scroll that produces it. Doing this amounts to setting $y(t) = k(y_0(t))$, which in turn gives the functional equation

$$k(h(t)) = -g(-1, h(t), t),$$

where we assume that k is a known function of a single variable, and h and g are to be determined so that the result defines the function f in the above Lemma and then the functions f and g together must define the flow from which the single scroll is derived. We leave this line of inquiry as an open question and continue development of the definition of the type-I generalized Chua equations.

2.7. *Formal definition of tupe-I generalizations of Chua's equations*

The single scroll construction can be carried out with any two-dimensional flow which always crosses the line $x = 1$ given an initial condition on the line $x = -1$. Such flows are easy to construct and we illustrate this by using two familiar equations to construct two examples of new double scrolls.

Example 1. For the first example we use the following variation on Duffing's equation:

$$\ddot{x} - s\dot{x} + x^3 = 0,$$

where $s > 0$ rather than $s \leq 0$ which usually defines Duffing's equation. The result of choosing s positive is to make the critical point of the equation a source rather than a sink.

We rewrite Duffing's equation in a matrix form with the critical point translated to the point (a, b):

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{bmatrix} 0.0 & -1.0 \\ (x-a)^2 & s \end{bmatrix} \begin{pmatrix} x-a \\ y-b \end{pmatrix}. \qquad (19)$$

The type-I generalized Chua equation from which we may obtain a double scroll based on this variation on Duffing's equation is given by

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{bmatrix} 0 & -1.0 & 0.0 \\ (x - a\,\mathrm{sgn}(u))^2 & s & 0.0 \\ 0.0 & 0.0 & -\gamma \end{bmatrix} \times \begin{pmatrix} x - a\,\mathrm{sgn}(u) \\ y - b\,\mathrm{sgn}(u) \\ z - \mathrm{sgn}(u) \end{pmatrix}, \qquad (20)$$

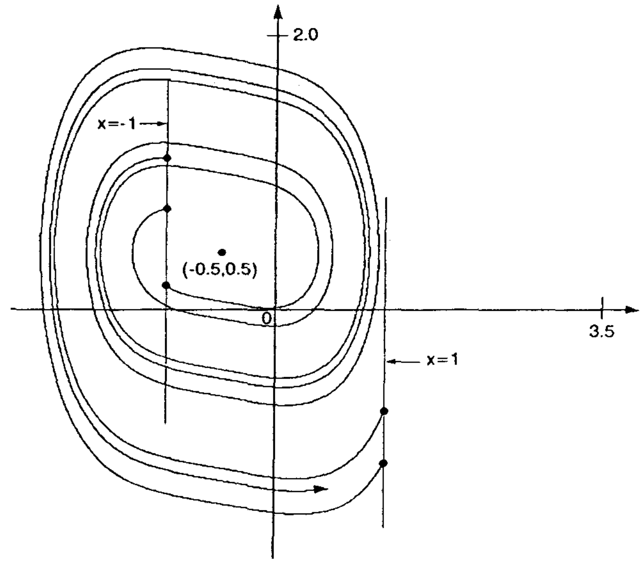

Fig. 7(a). Single scroll using the Duffing oscillator for the two-dimensional vector field.

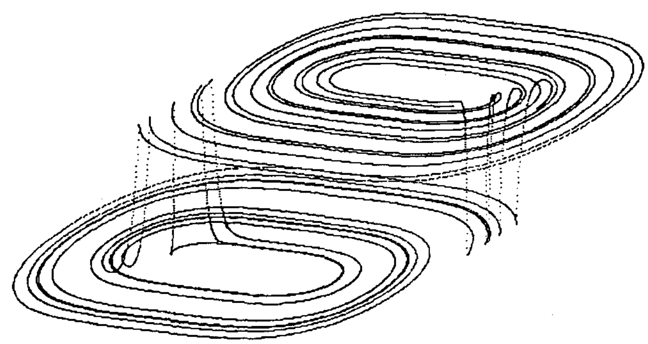

Fig. 7(b). Double scroll for the type-I generalized Chua equations using the Duffing oscillator as the two-dimensional vector field for the two-dimensional single scroll.

where a, b, s, and u are the same as for Eq. (16). Figure 7(a) is the two-dimensional single scroll determined by Eq. (19). Figure 7(b) is the double scroll produced by Eq. (20). In these figures $a = -0.5$, $b = 0.5$, $s = 0.17$, $\gamma = 100.0$. The initial conditions are $(-1.0, 0.1, 1.0)$.

Example 2. What we have done with this variation on Duffing's equation we may do with the van der Pol equation. The translated van der Pol equation in matrix form is given by

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{bmatrix} 0.0 & -1.0 \\ 1.0 & s(1.2-(x-a)^2) \end{bmatrix} \begin{pmatrix} x-a \\ y-b \end{pmatrix}. \qquad (21)$$

The type-I generalized Chua equation from which we may obtain a double scroll based on the van der Pol equation is given by

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{bmatrix} 0 & -1.0 & 0.0 \\ 1.0 & s(1.2-(x-a\,\mathrm{sgn}(u))^2) & 0.0 \\ 0.0 & 0.0 & -\gamma \end{bmatrix} \times \begin{pmatrix} x-a\,\mathrm{sgn}(u) \\ y-b\,\mathrm{sgn}(u) \\ z-\mathrm{sgn}(u) \end{pmatrix}, \tag{22}$$

where a, b, s, and u use the same as for Eq. (16). Figure 8(a) is the van der Pol single scroll from Eq. (21). Figure 8(b) is the associated double scroll from Eq. (22). In these figures $a = -0.5$, $b = 0.5$, $s = 0.43$, $\gamma = 10.0$. The initial conditions are $(-1.0, 0.1, 1.0)$.

Based on these examples and the work of Misiurewicz we formally define the type-I generalized Chua equations.

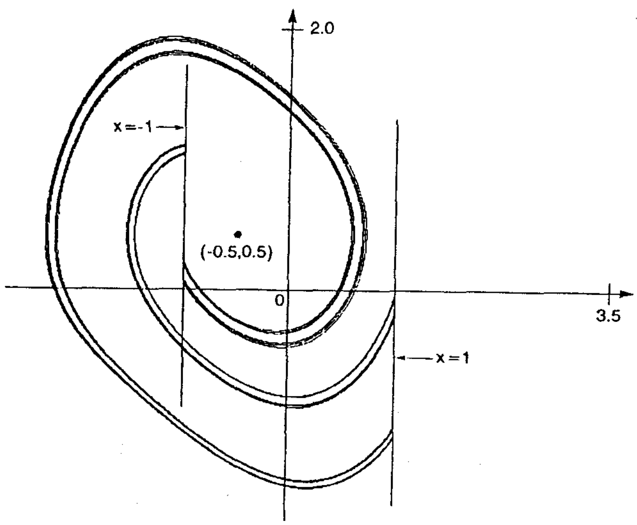

Fig. 8(a). Single scroll using the van der Pol oscillator for the two-dimensional vector field.

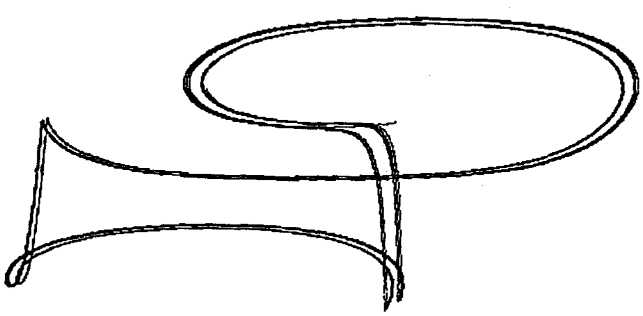

Fig. 8(b). Double scroll for the type-I generalized Chua equations using the van der Pol oscillator as the two-dimensional vector field for the two-dimensional single scroll.

Definition (Type-I Generalized Chua Equations). An equation of the form

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{bmatrix} a_{11}(x,y) & a_{12}(x,y) & 0.0 \\ a_{21}(x,y) & a_{22}(x,y) & 0.0 \\ 0.0 & 0.0 & -\gamma \end{bmatrix} \times \begin{pmatrix} x-a\,\mathrm{sgn}(u) \\ y-b\,\mathrm{sgn}(u) \\ z-\mathrm{sgn}(u) \end{pmatrix}, \tag{I}$$

where a, b, s are any real numbers and $u = z - x$ and $a_{ij}(x, y)$ are chosen so that the vector field

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{bmatrix} a_{11}(x,y) & a_{12}(x,y) \\ a_{21}(x,y) & a_{22}(x,y) \end{bmatrix} \begin{pmatrix} x-a \\ y-b \end{pmatrix}$$

defines a single scroll in $\mathbf{R}^2$, which will be called a type-I generalized Chua equation.

One of the most interesting applications of this analysis is suggested by Perez-Munuzuri [1992]. In that paper, a one-dimensional array of Chua circuits is coupled through its linear resistors and is used to study traveling wave fronts. Our analysis shows how to construct a one-dimensional array of type-I generalized Chua circuits coupled, mathematically, by using the x and y output of one equation as an input to change the constant γ in another equation. The constant affects the rate at which the solution of a type-I generalized Chua equation transitions between the two scrolls. In this sense, it is like a spring constant. The effect of one circuit on another will therefore occur mainly during the transition between scrolls. What is significant about our analysis is that it shows that each type-I generalized Chua circuit in the array can be based on a different two-dimensional flow depending on other design considerations. Such a one-dimensional array may have a very large range of dynamics.

2.8. *Poincaré maps for type-I generalized Chua equations*

In this subsection we provide an example to show how to generate Poincaré maps for type-I generalized Chua equations. We will use Eq. (16) for this purpose.

Equation (16) can be solved in closed form within each of the two regions in which it is linear. The analysis of the single scroll in subsection 2.2 shows that the Poincaré map is of the form FTFT where $\mathrm{T}(\mathbf{x}_0)$ is defined as the point where the solution of Eq. (16) (where $z > x$) reaches the plane

$z = x$. As in Brown & Chua [1991] we need only consider the map FT. Since F is the flip map, we need only examine the map T to which we now turn our attention.

For $z \geq x$ Eq. (16) can be solved in closed form:

$$x(t) = \exp(st)[(x_0 - a)\cos(t) - (y_0 - b)\sin(t)] + a,$$
$$y(t) = \exp(st)[(x_0 - a)\sin(t) + (y_0 - b)\cos(t)] + b,$$
$$z(t) = 1 + (z_0 - 1)\exp(-\gamma t)\,. \tag{23}$$

Since we start and end on the plane $z = x$ these equations simplify slightly. First we may take $z_0 = x_0$. Second, at some time later, we need $z(\tau) = x(\tau)$. In order for this to happen while $z > x$ the starting point for the use of Eq. (23) must have $x \leq 0.0$ and $y \geq 0.0$, otherwise we immediately pass over into the region where $z < x$. Given these restrictions we must have:

$$x(\tau) = \exp(s\tau)[(x_0 - a)\cos(\tau) - (y_0 - b)\sin(\tau)] + a,$$
$$y(\tau) = \exp(s\tau)[(x_0 - a)\sin(\tau) + (y_0 - b)\cos(\tau)] + b,$$
$$z(\tau) = 1 + (x_0 - 1)\exp(-\gamma\tau)\,. \tag{24}$$

Upon reaching the place $z = x$ at time τ, Eq. (24) must hold. We then apply the flip map F which maps this plane onto itself, and repeat the process. This process produces the map FT and FTFT is the Poincaré map. Hence Eq. (24) combined with the flip map produces the Poincaré map for Eq. (16).

The function $u = H(x, y, z)$ in Eq. (16) provides a natural first return surface on which to define the Poincaré map when we set $u = 0$. This is how the plane $z = x$ was obtained, since $u = z - x$. Clearly the map F is not complex, and the map T defined by Eq. (23) is not either, but together along with the plane $z = x$ the Poincaré map becomes very complex, i.e., chaotic. As $\gamma \to \infty, x(\tau) \to 1$ and Eq. (24) reduce to

$$1 = \exp(s\tau)[(x_0 - a)\cos(\tau) - (y_0 - b)\sin(\tau)] + a, \tag{25.1}$$
$$y(\tau) = \exp(s\tau)[(x_0 - a)\sin(\tau) + (y_0 - b)\cos(\tau)] + b. \tag{25.2}$$

Also, since the flip must map $(1, y(\tau))$ to $(-1, -y(\tau))$ we must have $x_0 = -1$ after one iteration. From this we obtain the equations

$$1 = \exp(s\tau)[(-1 - a)\cos(\tau) - (y_0 - b)\sin(\tau)] + a, \tag{26.1}$$
$$y(\tau) = \exp(s\tau)[(-1 - a)\sin(\tau) + (y_0 - b)\cos(\tau)] + b. \tag{26.2}$$

from which we conclude that the graph of the Poincaré map (a one-dimensional map) for the two-dimensional single scroll can be obtained in parametric equations with the parameter τ. Specifically, we may solve Eq. (26.1) for y_0 as a function of τ and substitute this into Eq. (26.2) to obtain an expression for $y(\tau)$ as a function of τ. In order to obtain the Poincaré map explicitly we would have to solve Eq. (26.1) for τ as a function of y_0, and then substitute this value of τ into Eq. (26.2). This will be very difficult. This parametric representation allows us to compute the derivative of $y(\tau)$ with respect to y_0 and thus provides valuable information about the Poincaré map.

When $\gamma < \infty$ we are still able to derive information about the Poincaré map for a three-dimensional system from the two-dimensional system. If the three-dimensional system is to have a fixed point then the time of transit from the initial condition to the final condition for the z variable, which must be $-x_0$, is given by

$$\tau = \log((1 - x_0)/(1 + x_0))/\gamma. \tag{27.0}$$

From this we conclude that the fixed points of the three-dimensional Poincaré map must be such that $|x| < 1$, and, even for γ small, x_0 must be near -1.0. The fixed points of Eq. (16) may be obtained from the following equations by trial and error:

$$x(\tau) = \exp(s\tau)[(x_0 - a)\cos(\tau) - (y_0 - b)\sin(\tau)] + a, \tag{27.1a}$$
$$y(\tau) = \exp(s\tau)[(x_0 - a)\sin(\tau) + (y_0 - b)\cos(\tau)] + b, \tag{27.1b}$$
$$-x_0 = 1 + (x_0 - 1)\exp(-\gamma\tau)\,. \tag{27.1c}$$

where (x_0, y_0) is a fixed point if $x(\tau) = -x_0$ and $y(\tau) = -y_0$.

Equation (27.1c) gives the value of τ for a fixed point of the Poincaré map of Eq. (16). With this value of τ we may use Eqs. (27.1a) and (27.1b) to determine if a fixed point has been obtained.

We can do better still. If (x_0, y_0, x_0) is a fixed point of FT then the following relation holds:

$$\begin{pmatrix} -x_0 - a \\ -y_0 - b \end{pmatrix} = \exp(s\tau) \begin{bmatrix} \cos(\tau) & -\sin(\tau) \\ \sin(\tau) & \cos(\tau) \end{bmatrix} \times \begin{pmatrix} x - a \\ y - b \end{pmatrix}, \tag{27.2}$$

where τ is determined by Eq. (27.0). This is an equality between two vectors so that, following Brown & Chua [1991], the square of the lengths of these two vectors must also be equal. After substituting for τ and using this fact we obtain the formula

$$\left(\frac{1 - x_0}{1 + x_0}\right)^{2s/\gamma} = \frac{(x_0 + a)^2 + (y_0 + b)^2}{(x_0 - a)^2 + (y_0 - b)^2}, \tag{27.3}$$

which reduces to the following quadratic equation in y_0,

$$y_0^2 + 2b\left(\frac{1 + r}{1 - r}\right) y_0 + b^2 + R/(r - 1) = 0,$$

where

$$r = \left(\frac{1 - x_0}{1 + x_0}\right)^{2s/\gamma}$$

and

$$R = r(x_0 - a)^2 - (x_0 + a)^2.$$

This equation states that the fixed points lie on the curve defined by Eq. (27.3). The value of Eq. (27.3) is that it reduces finding a fixed point to a trial and error search for x_0 only, y_0 being determined by x_0 through Eq. (27.3) and $z_0 = x_0$. We caution the reader that Eq. (27.3) is a necessary condition for a point in the plane $z = x$ to be fixed, but is not a sufficient condition. If we start with Eq. (27.2) we can solve for $\exp(s\tau)\cos(\tau)$ and $\exp(s\tau)\sin(\tau)$ in terms of x_0, y_0 and then obtain an expression for $\tan(\tau)$ where τ is the time of transit from Eq. (27.0). Doing this we obtain a second curve given by the equation

$$\tan(\tau) = \frac{2(x_0 b - y_0 a)}{x_0^2 + y_0^2 - (a^2 + b^2)}. \tag{27.4}$$

The intersection of Eqs. (27.3)–(27.4) determine necessary and sufficient conditions for a point (x_0, y_0, x_0) to be fixed by FT.

We anticipate that Eq. (16) may be used to derive an analytical test for chaos similar to the twist-and-flip map in Brown & Chua [1991]. Consequently, a more detailed analysis of the Poincaré map for Eq. (16) will be given in a later paper.

3. Type-II Generalized Chua Equations and the Dynamics of Rössler and Lorenz

In the previous section we have seen one method for extending the Chua equations based on using two-dimensional flows as building blocks of double scrolls. In this section we use the analysis of Sec. 2.0 to form a generalization in an entirely different direction.

The role of the sigmoid function and the sgn function in leading to a type-I generalized Chua equation which can be analyzed in terms of a one-dimensional map suggests that there is another direction of generalization for which, unlike the type-I generalized Chua equations, the only nonlinearities are those of sigmoid, piecewise-linear, and sgn functions.

Definition (Type-II Generalized Chua Equations). Let the solutions of the following vector ODE be unique in a bounded region of $\mathbf{R}^n$:

$$\dot{\mathbf{x}} = \mathbf{A}(\mathbf{x})(\mathbf{x} - F(\mathbf{x})), \tag{II}$$

where $\mathbf{A}(\mathbf{x})$ is an $n \times n$ matrix function of $\mathbf{x}$, and F is a mapping of $\mathbf{R}^n$ to itself.

A type-II generalized Chua equation is an ODE of the above form in which the components of the matrix $\mathbf{A}$ and the vector function F are composed of finite linear combinations of sigmoid functions, piecewise-linear functions, or sgn functions.

In addition to the usefulness of this equation for analysis, there are numerical advantages for modeling and simulation in that we are able to generate simple maps that are easy to evaluate on a computer and which have a wide range of dynamics. In fact, from the following examples we may conclude that we are able to construct a type-II generalized Chua equation having almost any dynamics we desire.

The key to the following constructions is found in how we view the various parts of the type-II generalized Chua equation:

$$\dot{\mathbf{x}} = \mathbf{A}(\mathbf{x})(\mathbf{x} - F(\mathbf{x})). \tag{28}$$

In duplicating the dynamics of a given vector field, the matrix $\mathbf{A}$ can be chosen to be the linearization of the given vector field at the nonzero fixed points.

Hence if $\mathbf{x}_0$ is a nonzero fixed point, then the matrix is chosen to be the function of this point, $\mathbf{A}(\mathbf{x}_0)$, which is the linear part of the vector field at the fixed point $\mathbf{x}_0$. Similarly, F is chosen to be a function of the fixed points $F(x_0)$. For example, if there are just three fixed points, say $\mathbf{x}_0$, 0.0 and $-\mathbf{x}_0$ a single sigmoid or signum function will likely be sufficient to construct F.

When there are three fixed points, the fixed point at 0.0 need not play a direct role in the creation of chaos as shown by the analysis in Sec. 2. What is important is the presence of at least two fixed points. Since, at this time, there is only the beginnings of a theory for type-II generalized Chua equations, we will use two examples to illustrate a general method for the construction of a dynamical system. Our two examples will be the Rössler dynamical system and the Lorenz dynamical system. What are will do in the next subsection is to construct type-II generalized Chua equations having dynamics very similar to these two systems.

3.1. *Rössler dynamics from a type-II generalized Chua equation*

The Rössler dynamics can be obtained from a type-II generalized Chua equation as follows:

First we write down the Rössler equations

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{bmatrix} 0.0 & -1.0 & -1.0 \\ 1.0 & 0.398 & 0.0 \\ 0.0 & 0.0 & -4.0 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0.0 \\ 0.0 \\ 2.0 + xz \end{pmatrix} . \tag{29}$$

Next we determine the fixed points. There are two:

$$x_0 = 2 \pm 1.7899, \quad y_0 = -x_0/0.398, \quad z_0 = -y_0;$$

for reasons of convenience that will be clear shortly we write x_0 as

$$x_0 = 2 \pm \lambda .$$

Conveniently, the coordinates of the two fixed points of the Rössler equation can be expressed by a single parameter, $\text{sgn}(\lambda)$, the sign of λ.

Using the λ notation, the linear part of the vector field at these fixed points is given by

$$\mathbf{A} = \begin{bmatrix} 0.0 & -1.0 & -1.0 \\ 1.0 & 0.398 & 0.0 \\ (2 \pm \lambda)/0.398 & 0.0 & -2.0 \pm \lambda \end{bmatrix} .$$

The function $F(\mathbf{x})$ serves to define the fixed points of the type-II generalized Chua equation. As noted earlier it is determined by the fixed points of the Rössler equations. Hence F is given by

$$F(\mathbf{x}) = \begin{bmatrix} (2.0 \pm \lambda) \\ -(2.0 \pm \lambda)/0.398 \\ (2.0 \pm \lambda)/0.398 \end{bmatrix} ,$$

depending on λ.

One way of writing this so that the symbol "±" is removed is to use a sigmoid or sgn function. We choose the sigmoid function defined in Eq. (7). Doing this we rewrite F as

$$F(\mathbf{x}) = \begin{bmatrix} (2.0 + \lambda g(u)) \\ -(2.0 + \lambda g(u))/0.398 \\ (2.0 + \lambda g(u))/0.398 \end{bmatrix} ,$$

where $g(u)$ is as in Eq. (7) and u must now be chosen as a function of (x, y, z) so that the vector field will equal the linear part of the Rössler equation near each nonzero fixed point. In fact the equation we construct will be linear in two regions determined by the two fixed points of the Rössler equations. Unfortunately, the function u must be chosen as $u = x - y^2$, and we recognize that how we arrived at this choice is not straight forward. The analysis of Misiurewicz [1993] offers the best insight. But due to the elementary level of our present theory we cannot say more at this time. In the next example, the choice of u is more simple to determine. At a later time we anticipate that there will be a complete theory that determines u. The function u may be thought of as a transition function since it defines a surface in three-space which separates the two fixed points, and for which the equation we are constructing is a different linear equation within each region. In the region where $x > y^2$, the vector field is determined by the fixed point where $x = 2 + \lambda$. When $x < y^2$, the vector field is determined by $x = 2 - \lambda$. As with Eq. (16), this surface provides the two-dimensional surface on which a Poincaré map may be defined. In each of these regions, the matrix $\mathbf{A}$ is also determined by these same conditions. One way to write $\mathbf{A}$ in a formula without the use of symbol "±" is as follows:

$$\mathbf{A} = \begin{bmatrix} 0.0 & -1.0 & -1.0 \\ 1.0 & 0.398 & 0.0 \\ (2+\lambda g(u))/0.398 & 0.0 & -2.0+\lambda g(u) \end{bmatrix} .$$

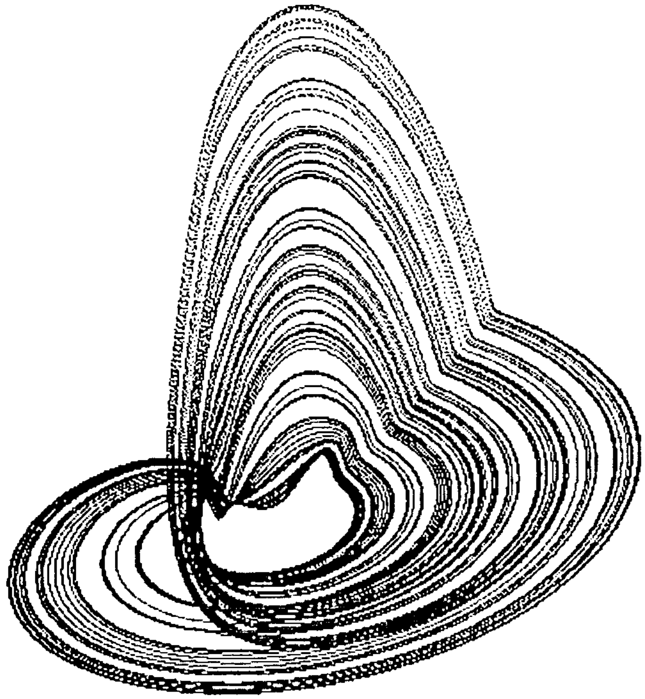

Fig. 9. The attractor for Rössler-like dynamics in a type-II generalized Chua equation.

Combining F and $\mathbf{A}$ according to **II** gives the desired type-II generalized Chua equation.

A key factor in this construction is to note that the linear part of the vector field varies with the fixed point, and hence we forced the matrix $\mathbf{A}$ to vary accordingly by use of the function $g(u)$, which, as before, is defined in Eq. (7). In doing this we made the matrix $\mathbf{A}$ function like the twist matrix used in the twist-and-flip map definition [Brown & Chua, 1991].

Figure 9 is the attractor for this equation, where $\gamma = 3.0$.

3.2. *The Lorenz dynamics from a type-II generalized Chua equation*

We now produce the Lorenz-like dynamics from a type-II generalized Chua equation. The Lorenz equations are

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{bmatrix} -10.0 & 10.0 & 0.0 \\ 28.0 & -1.0 & 0.0 \\ 0.0 & 0.0 & -2.67 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0.0 \\ -xz \\ xy \end{pmatrix} . \tag{30}$$

The three fixed points are approximately

$$x_0 = y_0 = \pm 8.48, \quad z_0 = 27.0,$$

and $(0, 0, 0)$.

The linear part of the vector field for a fixed point is given by

$$\mathbf{A} = \begin{bmatrix} -10.0 & 10.0 & 0.0 \\ 1.0 & -1.0 & x_0 \\ -x_0 & -x_0 & -2.67 \end{bmatrix},$$

and we choose F as

$$F(\mathbf{x}) = \begin{bmatrix} x_0\, g(x) \\ x_0\, g(x) \\ 27.0 \end{bmatrix}.$$

We now generate the nonlinear matrix $\mathbf{A}$ from the linear part of the vector field:

$$\mathbf{A} = \begin{bmatrix} -10.0 & 10.0 & 0.0 \\ 1.0 & -1.0 & x_0\, g(x) \\ -x_0\, g(x) & -x_0\, g(x) & -2.7 \end{bmatrix},$$

where, as in the Rössler map, $g(x)$ is given by Eq. (7), and $\gamma = 3.0$. Figure 10 is the attractor for this map. In this case we were able to take $u = x$ since the transition from one linear region to the other takes place when $x = 0$, that is the surface of transition is the $y - z$ plane, and provides the natural surface for defining the Poincaré map. It is routine to replace the sigmoid with the sgn function and obtain a completely piecewise linear Lorenz equation. In this equation, a piecewise linear one, it should be possible to find the one-dimensional map that describes its dynamics numerically, and very possibly analytically. It will be of interest to determine its similarity to the one-dimensional map obtained from the axiomatic Lorenz equations.

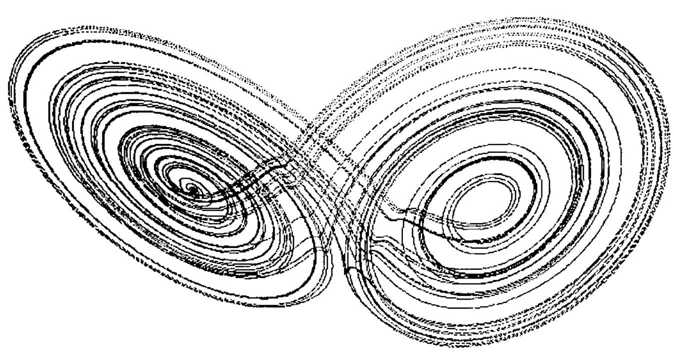

Fig. 10. The attractor for Lorenz-like dynamics in a type-II generalized Chua equation.

3.3. Partially decoupling a type-II generalized Chua equation

In subsection 2.1 we decoupled the type-I generalized Chua equation through a coordinate transformation. Doing this separated the stable and unstable eigenvectors and their corresponding manifolds, which in turn led to being able to reduce the type-I generalized Chua equation to a two-dimensional single scroll and then to a one-dimensional map. In a type-II generalized Chua equation we encounter some difficulties in doing this which we now examine.

Consider the type-II generalized Chua equation

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{bmatrix} -4.0 & 4.0 & 0.0 \\ 0.0 & -1.0 & -4.0\,\mathrm{sgn}(x) \\ 4.0\,\mathrm{sgn}(x) & 4.0\,\mathrm{sgn}(x) & -0.5 \end{bmatrix} \times \begin{pmatrix} x - \mathrm{sgn}(x) \\ y - \mathrm{sgn}(x) \\ z - 8 \end{pmatrix}.$$

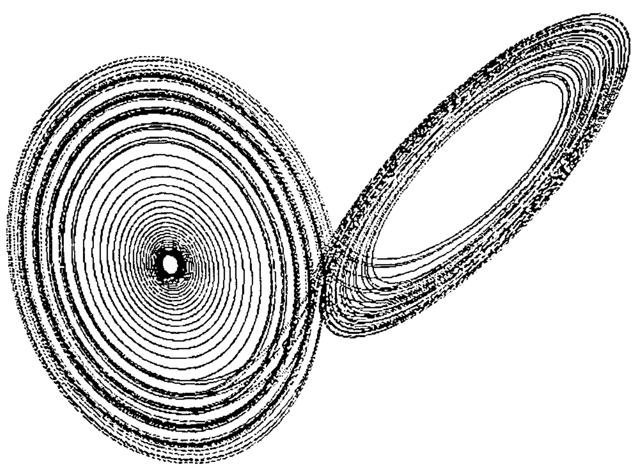

Fig. 11(a). The attractor for Lorenz-like dynamics in a type-II generalized Chua equation.

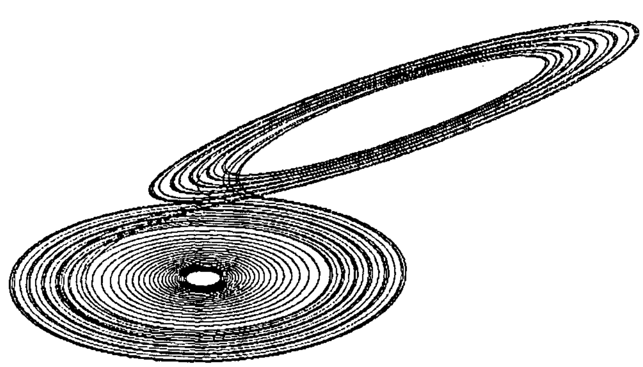

Fig. 11(b). The attractor for Lorenz-like dynamics in a partially decoupled type-II generalized Chua equation.

Figure 11(a) is the attractor for this equation and, as can be seen in this figure, it resembles a Lorenz attractor.

In analogy with the analysis of the twist-and-flip map found in Brown & Chua [1991], p. 405, we define two functions which will allow us to simplify the decoupling process:

$$\mathrm{sg}_1(u) = (1 + \mathrm{sgn}(u))/2,$$
$$\mathrm{sg}_2(u) = (1 - \mathrm{sgn}(u))/2.$$

Using these functions we are able to rewrite the above differential equation in the form

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = \mathrm{sg}_1(x) A_1 \begin{pmatrix} x - 1 \\ y - 1 \\ z - 8 \end{pmatrix} + \mathrm{sg}_2(x) A_2 \begin{pmatrix} x + 1 \\ y + 1 \\ z - 8 \end{pmatrix},$$

where,

$$A_1 = \begin{bmatrix} -4.0 & 4.0 & 0.0 \\ 0.0 & -1.0 & -4.0 \\ 4.0 & 4.0 & -0.5 \end{bmatrix},$$

and,

$$A_2 = \begin{bmatrix} -4.0 & 4.0 & 0.0 \\ 0.0 & -1.0 & 4.0 \\ -4.0 & -4.0 & -0.5 \end{bmatrix}.$$

At this point we may proceed with coordinate transformations which will decouple one of these matrices but will not generally decouple the other. Specifically, if we decouple A_1 we obtain a new equation

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = \mathrm{sg}_1(u) B_1 \begin{pmatrix} x - 3.34 \\ y - 0.187 \\ z + 1.97 \end{pmatrix} + \mathrm{sg}_2(u) B_2 \begin{pmatrix} x - 2.51 \\ y - 0.424 \\ z + 2.66 \end{pmatrix},$$

where

$$B_1 = \begin{bmatrix} 0.0 & -23.15 & 0.0 \\ 1.0 & 0.116 & 0.0 \\ 0.0 & 0.0 & -5.62 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} -1.81 & 12.77 & -2.4 \\ -0.89 & -0.83 & 0.18 \\ -1.96 & -37.8 & -2.86 \end{bmatrix},$$

and $u = x - 2y + z$. Figure 11(b) is the attractor in the new coordinates.

We will not attempt to reduce this equation to a one-dimensional map.

3.4. *A type-II generalized Chua equation in which both matrices are decoupled*

In this section we show that it is possible to write down examples of type-II generalized Chua equations in which the matrices A_1, A_2 are already decoupled and for which there exists a chaotic attractor. The specific form in which we will do this will be a variation of that used by Misiurewicz and cited earlier in subsection 2.4. Our totally decoupled equation is given by the following formulas:

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = \mathrm{sg}_1(x) A_1 \begin{pmatrix} x-1 \\ y \\ z-1 \end{pmatrix} + \mathrm{sg}_2(u) A_2 \begin{pmatrix} x-1 \\ y \\ z+0.5 \end{pmatrix},$$

where,

$$A_1 = \begin{bmatrix} 0.0 & 0.0 & -20.0 \\ 0.5 & -1.0 & 0.0 \\ 1.0 & 0.5 & 0.0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.1 & 4.0 & 0.0 \\ -4.0 & 0.1 & 0.0 \\ 0.0 & 0.0 & -1.0 \end{bmatrix},$$

and $u = x - y/4 + z$. Figure 12(a) is the attractor for this equation.

This equation seems to provide a good example to use for Shil'nikov's theorem. In particular, it should be possible to prove that this system (if we make the vector field smooth) satisfies the hypothesis of Shil'nikov's theorem without much difficulty. In order to make the vector field smooth we can replace the sgn function with the sigmoid function Eq. (7).

In particular if we use the sigmoid function Eq. (7) with $\gamma = 30.0$ in place of the sgn function in the above equation, we have the attractor seen in Fig. 12(b).

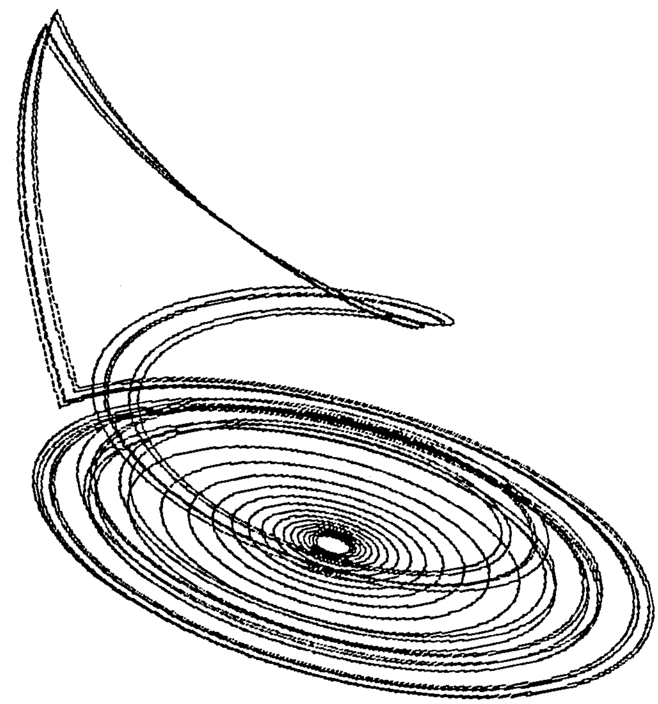

Fig. 12(a). The attractor for a totally decoupled type-II generalized Chua equation.

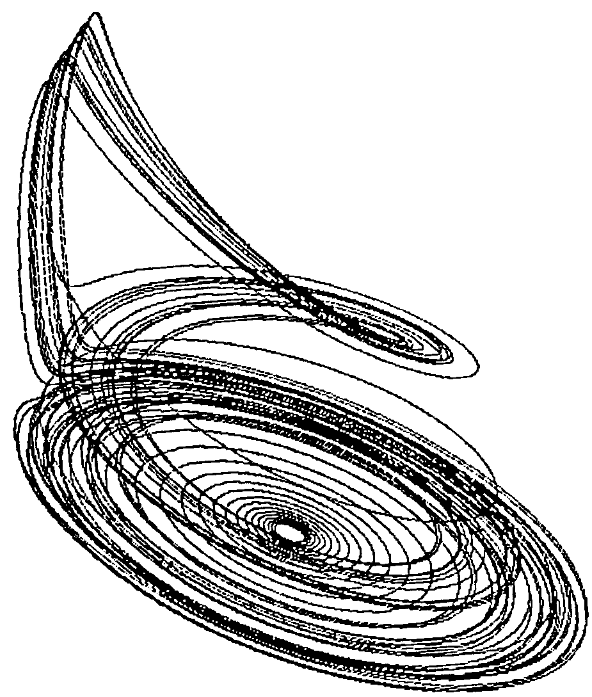

Fig. 12(b). The attractor for a totally decoupled type-II generalized Chua equation in which the SGN function has been replaced by a sigmoid function.

4. Summary and Conclusions

A complete understanding of the dynamics of the type-I and type-II generalized Chua equations would seem to be valuable in the study of chaos in general. It is likely that until these simple examples are fully understood and there exists a "complete" set of theorems relating the horseshoes to Liapunov exponents, to power spectral densities, to topological entropy, to one-dimensional maps and to fractals, our understanding of chaos will remain very meager and fragmented. For if we are unable to understand these, the simplest of all chaotic dynamical systems arising from autonomous ordinary differential equations, what can we understand of more complex systems?

In the same vein, if there is to emerge an era of new technology that utilizes chaotic dynamics to develop new concepts for computers, signal detectors, and controls to harness fundamentally unstable devices based on the structure of the nervous system, then we must be able to construct complex dynamical systems with predictable properties from simple, nonchaotic, and dynamically stable components. Understanding systems based on type-I and type-II generalized Chua equations and the twist-and-flip equations will provide a large step toward these technology goals.

The two-dimensional single scroll derived in Sec. 2.2 reveals that chaos in both type-I and type-II generalized Chua equations arises from how separate linear regions are pieced together. The example of Eq. (11) and more generally Eq. (16) shows that "incommensurate frequencies", which plays the key role in the formation of almost periodic functions from periodic functions, is not a necessary ingredient for chaos to exist in a dynamical system. What is important is the presence of hyperbolic dynamics which is all that is present in these examples. But although hyperbolic dynamics is the simplest ingredient necessary, it is still not sufficient to produce chaos even when the dynamics are bounded. In fact it is possible to select the parameters a, b, s, and γ in such a way that the attractor in Eq. (13) is periodic. This suggests that the type-I and type-II generalized Chua systems will be classified as chaotic or nonchaotic according to the manner in which the linear regions are "pieced together." Stuart Cowan [Cowan, 1992] in his Ph.D. dissertation demonstrates the power of the process of "piecing together" linear regions of a flow for obtaining chaos.

Further, our analysis suggests that this "piecing together" of separate linear regions defines one-dimensional maps, one map for each pair of adjoining regions. This observation reinforces the belief that a large class of high dimensional autonomous chaotic systems can be classified by one-dimensional maps. (However, the n-dimensional twist-and-flip map shows that this cannot, in general, be done for nonautonomous systems.)

In addition to the value of these maps in providing a road map to the development of a *theory of chaotic dynamical systems*, they also provide a guide to the design and development of a large array of chaotic circuits from simple and familiar circuit elements. The building block approach to the design of circuits that they provide also includes a building block approach to the development of dynamical models having predetermined chaotic dynamics. This will provide a much needed road map for researchers in the biological sciences to design complex models of biological dynamical systems from simple and well understood components.

The dynamic diversity of these systems as illustrated in their being able to reproduce the Rössler and Lorenz dynamics and their analytical relationship to the theory of one-dimensional maps, as well as their simplicity, seem to suggest that they are, in some sense, fundamental building blocks of autonomous chaotic systems and are thus the proper subjects of study directed toward a comprehensive and unifying understanding of chaos.

We have also shown that these two directions of research illustrate that the two-dimensional single scroll is a paradigm for chaos that is the autonomous analogue of the twist-and-flip paradigm [Brown & Chua, 1992], and we conclude that the twist-and-flip paradigm that arises from two-dimensional forced nonlinear oscillators and the autonomous twist-and-flip paradigm that arises from the type-I and type-II generalized Chua equations explain a large class of chaotic dynamical systems arising from ODEs.

Acknowledgments

We would like to thank Prof. Morris Hirsch for the many useful conversations and suggestions leading to the reduction of the simplified version of Chua's equation in Sec. 2.1 to a one-dimensional map.

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Appendices

A. *Construction of Generalized Chua Maps from Generalized Chua Equations*

Consider the type-II generalized Chua equations

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{bmatrix} -1.2857 & 9.0 & 0.0 \\ 1.0 & -1.0 & 1.0 \\ 0.0 & -14.2857 & 0.0 \end{bmatrix} \times \begin{pmatrix} x - a\,\mathrm{sgn}(x) \\ y \\ z \end{pmatrix}.$$

From the well-known theory of linear ODEs [4] we can solve this equation in closed form when $\mathrm{sgn}(x) = 1$ or when $\mathrm{sgn}(x) = -1$. To find a map for the type-I or type-II generalized Chua equations we must combine these two solutions into one equation in some manner. The obstacle to combining them is that each is separately a function of time, and we must switch between the two equations as the solution crosses between the regions where x is positive and where x is negative. This can be done if we use small steps and check at each step to see if x has changed sign. In order to see how to construct such a function we first consider the linear vector ODE

$$\dot{\mathbf{x}}(t) = A(\mathbf{x} - a\,\mathrm{sgn}(x)\mathrm{e}_1)\,,$$

where A is 3×3 matrix, $\mathbf{x}$ is three-dimensional column vector defined by

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

and $\mathbf{e}_1$ is the column vector $(1.0, 0.0, 0.0)^T$. The solution to this equation is given by

$$\mathbf{x}(t) = \exp(At)(x_0 - a\,\mathrm{sgn}(x_0)\mathrm{e}_1) + a\,\mathrm{sgn}(x_0)\mathrm{e}_1.$$

If, in the above ODE. we replace the function $\mathrm{sgn}(x)$ by $f(x)$ where

$$f(x) = \frac{\exp(\gamma x) - 1.0}{\exp(\gamma x) + 1.0}$$

and γ is very large, the solution to

$$\dot{\mathbf{x}}(t) = A(\mathbf{x} - af(x)\mathrm{e}_1) \tag{2}$$

will be, dynamically, very close to

$$\mathbf{x}(t) = \exp(At)(\mathbf{x}_0 - a\,f(x_0)\mathrm{e}_1) + a\,f(x_0)\mathrm{e}_1, \tag{3}$$

although over a long time span a solution of Eq. (2) starting from $\mathbf{x}_0$ will diverge from the function $\mathbf{x}$ defined in Eq. (3); their qualitative behavior will be the same.

This observation leads us to seek a map in the form

$$\mathrm{T}(\mathbf{x}) = \exp(A\tau)(\mathbf{x} - F(\mathbf{x})) + F(\mathbf{x}).$$

If we choose τ to be small then there will be no problem when the solution moves between the various linear regions. For the equation given at the beginning of this appendix, a first approximation of the desired map is given by

$$\mathrm{T}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \exp\left(\tau \begin{bmatrix} -1.2857 & 9.0 & 0.0 \\ 1.0 & -1.0 & 1.0 \\ 0.0 & -14.2857 & 0.0 \end{bmatrix}\right) \times \begin{pmatrix} x - a\,\mathrm{sgn}(x) \\ y \\ z \end{pmatrix} + \begin{pmatrix} a\,\mathrm{sgn}(x) \\ y \\ z \end{pmatrix}.$$

Additional simplification is possible: For small $\tau, \exp(\tau A) \approx I + \tau A$, and so we seek a new matrix A based on the old matrix A above for which the map

$$\begin{aligned} \mathrm{T}(\mathbf{x}) &= (I + \tau A)(\mathbf{x} - F(\mathbf{x})) + F(\mathbf{x}) \\ &= \mathbf{x} + \tau A(\mathbf{x} - F(\mathbf{x})) \end{aligned}$$

will have dynamics similar to the Chua equation. At this point our procedure will have derived a map that agrees with the well-known forward, one-step Euler map. But in general, the Euler map may not have the same dynamics as the flow from which it was derived. We can fix this by making a small correction to the map that forces the determinant of the Jocobian of T to be

$$\exp(\mathrm{tr}(\mathbf{A}))\,.$$

We do not present the theory of such maps here but rather present a simple two-dimensional example to demonstrate the technique.

Example. Consider the equation

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ a - x^3 \end{pmatrix},$$

which is related to Duffing's equation. The Euler map for this equation is given by

$$\mathrm{T}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + \tau y \\ y + \tau(a - x^3) \end{pmatrix}.$$

The determinant of the Jacobian of this map is $1 + 3\tau^2 x^2$ which is greater than 1.0 everywhere except at 0.0. However, the flow that solves the ODE from which we derived this map is measure-preserving. We can make the map T measure-preserving without altering its relationship with the flow by adding a small perturbation to the Euler map as follows. Let

$$\mathrm{T}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + \tau(y + g(x)) \\ y + \tau(a - x^3) \end{pmatrix},$$

then we require that det(D(T))=1.0 and solve for $g(x)$. Doing this we get the new map

$$\mathrm{T}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + \tau(y - \tau x^3) \\ y + \tau(a - x^3) \end{pmatrix}.$$

This technique suggests that there is a class of simple closed form maps that generate the double scroll and are as easy to implement on a computer as the Euler, forward, one step integrator, i.e. the Euler map.

For example, the following type-I generalized Chua map produces a double scroll:

$$\mathrm{T}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \tau \begin{bmatrix} -1.0 & 2.0 & 0.0 \\ 1.0 & -1.0 & 1.0 \\ 0.0 & -1.0 & 0.0 \end{bmatrix} \times \begin{pmatrix} x - af(x) \\ y \\ z \end{pmatrix} + \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

A double scroll can be generated from this map by taking $a = 1.0, \gamma = 5.0$, and $\tau = 0.02$. It is easily seen that this map has three fixed points given by the functional equation $x = f(x)$ and the conditions $y = z = 0.0$. The solutions of the functional equa tion $x = f(x)$ are $x = 0.0$, and $x \approx \pm a$. At the fixed points where $x \approx (a,\ 0,\ 0)$ and $(-a,\ 0,\ 0)$ there is single eigenvalue less than 1.0 and a pair of complex conjugate eigenvalues with real parts greater than 1.0. This is dynamically identical to the Chua circuit. At $(0,0,0)$ the dynamics are also the same as the Chua circuit: the real eigenvalue is greater than one and the real part of the complex eigenvalues is less than 1.0.

We can obtain a Rössler-like band map from a generalized Chua map by a slight change to the matrix A. Making this change we have a simulated Rössler map

$$\mathrm{T}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \tau \begin{bmatrix} -1.0 & 2.0 & 0.0 \\ 0.6 & -1.5 & 1.0 \\ 0.0 & -1.0 & 0.0 \end{bmatrix} \times \begin{pmatrix} x - af(x) \\ y \\ z \end{pmatrix} + \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

We stress that these generalized Chua maps are not exactly the same as the Rössler and the Lorenz systems since the matrix $\mathbf{A}$ and the function F in the generalized Chua maps are composed of finite linear combinations of sigmoid functions, piecewise-linear functions, or sgn functions. However, their dynamics are similar.

Parallel to the Generalized Chua Equation defined earlier,

$$\dot{\mathbf{x}}(t) = \mathbf{A}(\mathbf{x})(\mathbf{x} - F(\mathbf{x}))\,,$$

we now define the Generalized Chua map:

Definition. The generalized Chua map will be any map of the form

$$\mathrm{T}(\mathbf{x}) = \mathbf{x} + \tau\mathbf{A}(\mathbf{x})(\mathbf{x} - F\mathbf{x}))\,.$$

B. *Some Properties of the Generalized Chua Map*

We now make some observations on the generalized Chua map T. T can be factored as

$$\mathrm{T}(\mathbf{x}) = \mathrm{L}_{F(\mathbf{x})}[(I + \tau A(\mathbf{x}))\mathrm{L}^{-1}_{F(\mathbf{x})}(\mathbf{x})]\,,$$

where $L_{F(\mathbf{x})}(W) = W + F(\mathbf{x})$. This means that the map $L_{F(\mathbf{x})}$ is a nonconstant translation map. When the nonlinearities in F are finite linear combinations of sgn functions only, T is easily inverted:

$$\mathrm{T}^{-1}(\mathbf{x}) = \mathrm{L}_{F(\mathbf{x})}(I + \tau A(\mathbf{x}))^{-1}\mathrm{L}^{-1}_{F(\mathbf{x})}(\mathbf{x})\,.$$

In any case, the fixed points of T can Now be easily deduced from

$$\mathrm{T}(\mathbf{x}) = \mathbf{x} + \tau A(\mathbf{x})(\mathbf{x} - F(\mathbf{x}))\,.$$

Since if $\mathrm{T}(\mathbf{x}) = \mathbf{x}$ then $\tau A(\mathbf{x})(\mathbf{x} - F(\mathbf{x})) = 0$ and so $F(\mathbf{x}) = \mathbf{x}$ when A is invertible. In other words, the fixed points of T are the fixed points of F for invertible matrices A.

The derivative of the map T, DT, is as follows:

$$\mathrm{DT} = I + \tau\left[\left(\frac{\partial A}{\partial x}\right)(\mathbf{x} - F(\mathbf{x}))\,,\right.$$
$$\left.\times\left(\frac{\partial A}{\partial y} + (\mathbf{x} - F(\mathbf{x})\right)\,,\,\left(\frac{\partial A}{\partial z}\right) + (\mathbf{x} - F(\mathbf{x}))\right]$$
$$+\,\tau\left[A\frac{\partial(\mathbf{x} - F(\mathbf{x}))}{\partial x}\,,\right.$$
$$\left.\times\, A\frac{\partial(\mathbf{x} - F(\mathbf{x}))}{\partial y},\, A\frac{\partial(\mathbf{x} - F(\mathbf{x}))}{\partial z}\right],$$

At the fixed points of T we have $F(\mathbf{x}) = \mathbf{x}$ so that the derivative simplifies

$$\mathrm{DT} = I + \tau\left[A\frac{\partial(\mathbf{x} - F(\mathbf{x}))}{\partial x}\,,\right.$$
$$\left.\times\, A\frac{\partial(\mathbf{x} - F(\mathbf{x}))}{\partial y},\, A\frac{\partial(\mathbf{x} - F(\mathbf{x}))}{\partial z}\right],$$

and this reduces to

$$\mathrm{DT} = \tau A(\mathbf{x})(I - \mathrm{D}F(\mathbf{x}))\,.$$

The simplest case of the map T is obtained when the matrix A is a constant function of $\mathbf{x}$.

C. *Construction of General Maps having Complex Dynamics*

Suppose we have divided all of $\mathbf{R}^3$ into cubes having a volume of one unit. This is a countable set and therefore the cubes can be labeled as C_1, C_2, $C_3, \ldots$ Now assume that within each cube we define a *local* vector field by a differential equation which is linear inside that cube and zero outside of the cube. Then the collection of all cubes and local vector fields define a vector field on all of three-dimensional space. This method of defining a vector field is a direct generalization of the Chua circuit equations. (S. Cowan uses a similar idea in this dissertation [Cowan, S., 1992]). It is possible to construct such vector fields explicitly using only sgn functions and to make these fields smooth by replacing the SGN function by a sigmoid function as we did in Sec. 3.4. The examples presented in this paper demonstrate just how complex the dynamics of such vector fields can be. In any case a general vector field can be constructed by this method and would have the form

$$\dot{\mathbf{x}} = \sum_{i=-\infty}^{+\infty} 1_{C_i}(\mathbf{x})A_i(\mathbf{x} - \mathbf{x}_i)\,,$$

where 1_{C_i} is the indicator or characteristic function of the cube C_i, A_i are matrices, and x_i are fixed vectors. One must be careful not to overlook the correct definition of this vector field on the boundaries of the cubes, but in numerical simulations the definition of the vector field on the boundaries of the cubes can generally be ignored due to the fact that integration routines have a finite step size.