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STRETCHING AND FOLDING TRANSITIONS IN THE HÉNON INFINITESIMAL DIFFEOMORPHISM

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Abstract

In [1] It was noted that, in the Smale horseshoe, the degree of stretching and folding was fixed. However, as also noted in [2,3,4] there is no reason to require stretching and folding to be only have a fixed value. In this is paper, we will explore how the dynamics of the Hénon ID vary with the duration of stretching and folding by varying the "step size" parameter in the ID of the Hénon map over a range of 0.00001 to 1. Additionally, the Hénon ID provides an entirely different view of the transition to chaos from the traditional period doubling route. Through the increase in the duration of stretching and folding, what the analysis shows is, that in place of period doubling, catastrophic transitions occur that may more accurately represent what we see in nature.

Keywords. Chaos, natural science, complexity, dynamical synthesis.

AMS (MOS) subject classification: 37D45.

1 Introduction

There is little doubt that stretching and folding is the force behind complex dynamics [1]. However, in contrast to its first use in proving a very difficult mathematical theorem, stretching and folding in the natural world need not be uniform [2,3,4]. It may occur in very small increments such as occurs in physics or in very large increments such as occurs in weather. The amount or duration of stretching and folding need not be equal either. This is also illustrated by weather. In general, by recognizing that the duration and combination of stretching and folding drives complexity, it may be possible to improve our ability to predict complex dynamics generally.

In this paper we examine the Hénon ID in which stretching and folding will be equal and represented by the ID parameter h. Unequal stretching and folding will be treated in a later paper.

The point of this paper is to introduce the concept of using stretching and folding in varying durations to obtain a wider range of dynamics and to understand how dynamics can evolve as stretching and folding varies or evolves. This is particularly important for natural (weather, geology, biology, etc) systems and human systems in that it may significantly improve cause and effect prediction in systems where presently there is only statistical correlation. Additionally, the use of IDs for modeling the variation of stretching and folding may significantly reduce the computational complexity of biological, human and weather models and thus advance these sciences.

An entirely different view of the ”transition to chaos” is presented by the Hénon ID and this view provides an interesting contrast to the traditional transition to chaos by period doubling. Through the increase in the duration of stretching and folding, what the analysis shows is, that in place of period doubling, catastrophic transitions occur that may more accurately represent what we see in nature.

2 The Hénon ID

As noted in [3,5] the Hénon ID is composed from three elementary IDs.

$$\mathbf{T}_1(h)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0.5\begin{pmatrix} 1+\cos(\pi h) & 1-\cos(\pi h) & \sqrt{2}\sin(\pi h) \\ 1-\cos(\pi h) & 1+\cos(\pi h) & -\sqrt{2}\sin(\pi h) \\ -\sqrt{2}\sin(\pi h) & \sqrt{2}\sin(\pi h) & 2\cos(\pi h) \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\mathbf{T}_2(h)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+(1-ay^2)h \\ y \\ z \end{pmatrix}$$

$$\mathbf{T}_3(h)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ \exp(\alpha h)y \\ z \end{pmatrix}$$

The composition of these three IDs is the Hénon ID. $\mathbf{T}_1, \mathbf{T}_3$ are folding and $\mathbf{T}_2$ is stretching [2]. The ID parameter h is the same for both stretching and folding. In the following section we will examine the impact of varying the degree of stretching and folding in the Hénon ID in order to gain insight into the potential complexities that arise from variable stretching and folding in dynamical systems generally.

3 The Transitions

The transitions will be presented in a series of figures in which the parameter h will vary from 0.00001 to 1.0. At $h = 1.0$, we will recover the traditional Hénon Attractor. In between, an interesting array of dynamics will appear.

The first study is for $h = 0.00001$, see Fig. 1 where we see only an attracting limit cycle. For $h \in (0.0, 0.5)$ a limit cycle persists and only becomes a stronger attractor as h increases. See Figs 1, 2, 3. At $h = 0.5$ a breakdown occurs in which the dynamics becomes manifested in a finite number of periodic points, see Fig 4 for convergence to a period 4 point. After reaching $h = 0.57$, see Fig 5, a distorted limit cycle reappears and evolves as h moves up to 0.599, see Fig 6. At $h = 0.6$ another breakdown occurs in which periodic points re-emerge; at $h = 0.79$ an unstable orbit shadows the Hénon Attractor Fig. 7; additional unstable dynamics also occur at other parameter values, example $h = 0.09$ or $h = 0.99$.

At $h = 0.8$, the first hint of the standard Hénon attractor emerges, see Fig. 8 and as we converge to $h = 1.0$, the classical Hénon attractor appears, see Fig. 9.

While there is much more to be explored in this study, we note that the variation of the duration of stretching and folding can have a far reaching impact on the dynamics of the underlying processes. This would explain why weather is so hard to predict: the stretching and folding forces in weather are themselves varying in their duration even though the underlying Newtonian equations are unchanged.

It would be impossible to constantly adjust the Newtonian equations to fit the changing forces of weather; however, the basic form of the IDs remain unchanged while the infinitesimal parameter alone changes. In particular, just changing the duration of stretching and folding can shift a system from a stable limit cycle to an unbounded hyperbolic system. Such an occurrence in nature could be disastrous and seemingly un-foreseeable unless we were specifically measuring the changes in stretching and folding duration of the system.

4 Summary

Stretching and folding is fundamental to complex processes, most commonly human and natural systems. Studying the variation in the duration of stretching and folding is made possible for systems modeled using IDs rather than ODEs. By studying the evolution of stretching and folding it may become possible to make predictions about the evolution of complex systems that are presently out of reach. Further, it is observed that by varying the duration of stretching and folding, the traditional period doubling route to chaos is replaced by catastrophic transitions between stable states that may more closely represent what we see in nature.

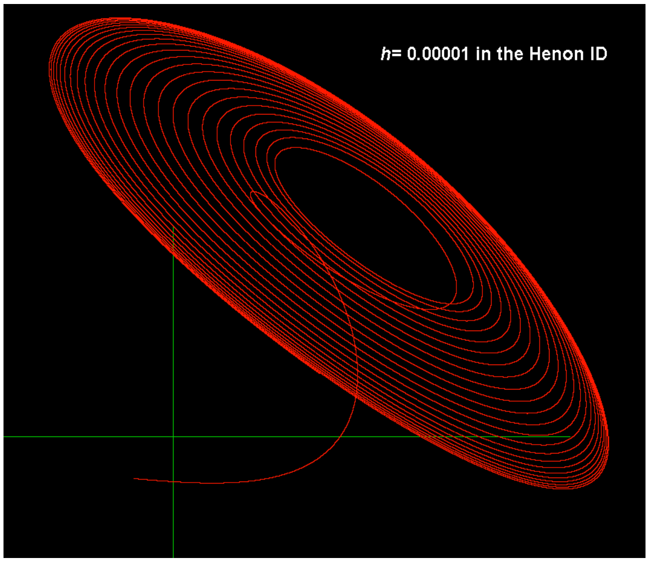

Figure 1: Trajectory of the Hénon ID for $h = 0.00001$

5 References

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[2] Brown, R. [2014] ”The Hirsch Conjecture”, to appear in *Dynamics of Continuous, Discrete and Impulsive Systems*

[3] Brown, R., [2015] ”The Theory of Infinitesimal Diffeomorphisms: An Introduction”, to appear.

[4] Brown, R. [2014] ”ISF Part II” to appear in DCDIS

[5] Brown, R., Chua, L. [1993] ”Dynamical Synthesis of Poincaré Maps”, *International Journal of Bifurcation and Chaos* **3**(5),1235-1267

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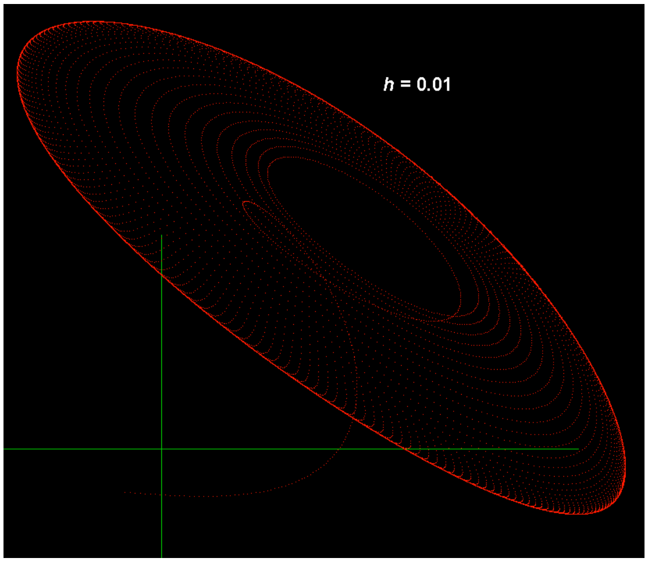

Figure 2: Trajectory of the Hénon ID for $h = 0.01$

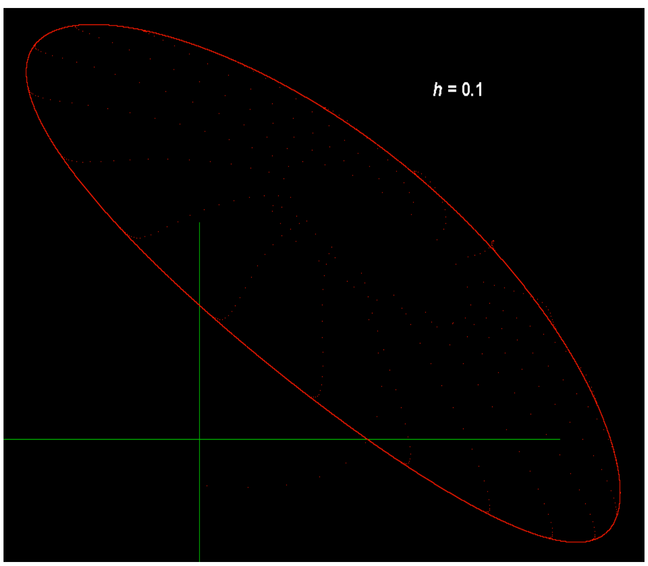

Figure 3: Trajectory of the Hénon ID for $h = 0.1$

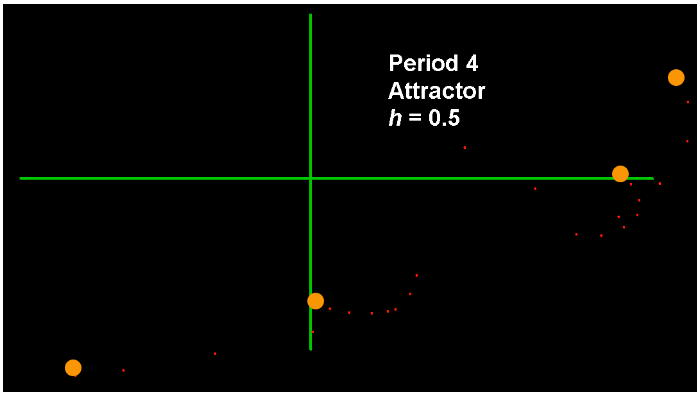

Figure 4: Period 4 Attractor for $h = 0.5$

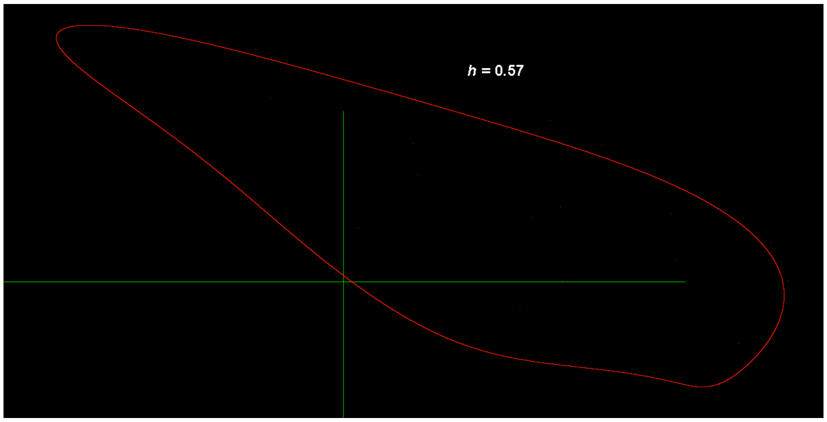

Figure 5: Trajectory of the Hénon ID for $h = 0.57$

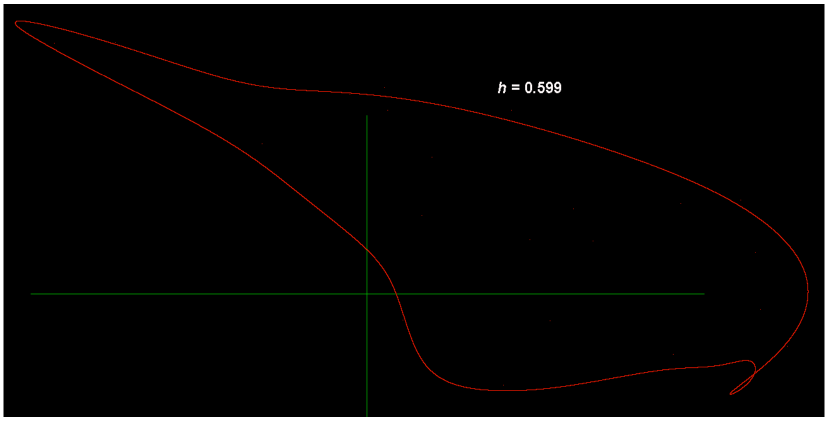

Figure 6: Trajectory of the Hénon ID for $h = 0.599$

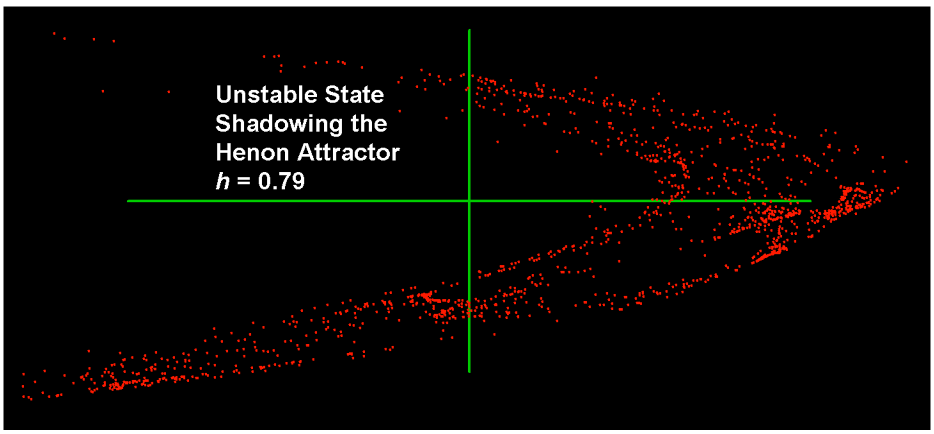

Figure 7: Unstable orbit Shadowing the Henon Attractor, $h = 0.79$

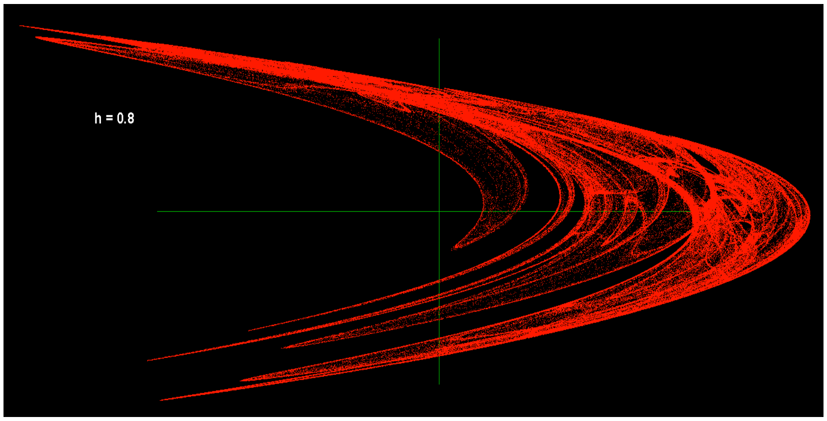

Figure 8: Trajectory of the Hénon ID for $h = 0.8$

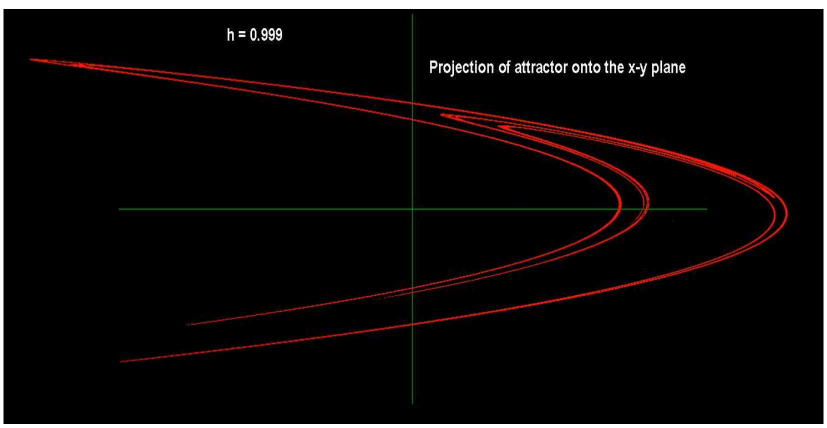

Figure 9: Trajectory of the Hénon ID for $h = 0.999$