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Horseshoes in the measure-preserving Hénon map

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Abstract. We show, using elementary methods, that for $0 < a$ the measure-preserving, orientation-preserving Hénon map, H, has a horseshoe. This improves on the result of Devaney and Nitecki who have shown that a horseshoe exists in this map for $a \geq 8$. For $a > 0$, we also prove the conjecture of Devaney that the first symmetric homoclinic point is transversal.

To obtain our results, we show that for a branch, C^u, of the unstable manifold of a hyperbolic fixed point of H, C^u crosses the line $y = -x$ and that this crossing is a homoclinic point, $\mathcal{X}_c$. This has been shown by Devaney, but we obtain the crossing using simpler methods. Next we show that if the crossing of $W^u(\mathbf{p})$ and $W^s(\mathbf{p})$ at $\mathcal{X}_c$ is degenerate then the slope of C^u at this crossing is one. Following this we show that if $\mathcal{X}_c$ is a degenerate homoclinic its x-coordinate must be greater than $1/(2a)$. We then derive a contradiction from this by showing that the slope of C^u at $\mathrm{H}^{-1}(\mathcal{X}_c)$ must be both positive and negative, thus we conclude that $\mathcal{X}_c$ is transversal.

Our approach uses a lemma that gives a recursive formula for the sign of curvature of the unstable manifold. This lemma, referred to as 'the curvature lemma', is the key to reducing the proof to elementary methods. A curvature lemma can be derived for a very broad array of maps making the applicability of these methods very general. Further, since curvature is the strongest differentiability feature needed in our proof, the methods work for maps of the plane which are only C^2.

1. *Introduction*

As a result of its simplicity, the Hénon map [8]

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} 1 + y - a\,x^2 \\ b\,x \end{pmatrix}$$

where a, b are parameters, has been the source of numerous investigations, both numerical and analytical. Among analytical studies is Benedicks and Carleson [1], who investigated the structure of strange attractors in the non-measure-preserving case, $b \neq \pm 1$. Devaney and Nitecki [6] treated both the measure-preserving and the non-measure-preserving cases. A result found in their paper is that if $b = \pm 1,\ a \geq 8$ then there is an embedded horseshoe in H. Devaney [5] has conjectured that the first symmetric homoclinic point of H is transversal. Ushiki [10] showed that analytic maps of the plane cannot have homoclinic loops. Combining this result with those of Devaney and Churchill and Rod [3] we can conclude there is some transverse homoclinic point for the Hénon map. However, this requires analyticity and does not prove Devaney's conjecture that the first symmetric homoclinic point is transversal. Although we give a proof for the Hénon map, $b = -1, a > 0$, that there is an embedded horseshoe, the method works for non-analytic maps as well. Thus, it is routine to show for $a > \pi/2$ the twist-and-flip map has an embedded horseshoe. By modifications of the Hénon map to make it only C^2, our proof still carries through.

A key technical tool used in this proof is the reversibility property of H. Devaney was the first to extend the notion of reversibility from flows to diffeomorphisms [4]. His aim in doing so was to utilize these powerful features of Hamiltonian systems exploited by De Volgelaere [7] in a more general setting. We retrace some of this development utilizing more elementary methods to obtain the existence of symmetric homoclinic points. A technical value of reversibility in our proof is that it allows us to confine our arguments to C^u without considering the path of C^s. Without these symmetries, we must develop a dual set of arguments to trace the paths of C^u and C^s until a transverse crossing is obtained. For example, if we assume $a > 1$, it is possible to obtain transverse homoclinic points without reversibility. The main significance of reversibility is that it allows us to obtain transversality over the entire parameter range of $a > 0$. Also, it is clearly essential to prove Devaney's conjecture.

In summary, our methods are distinct from the previous techniques used in that our analysis is based on knowing the sign of the curvature of $W^u(\mathbf{p})$; the proofs given are relatively simple and computational and can be directly applied to other maps; we obtain a result for the entire parameter range $a > 0$; and, the methods work for C^2 diffeomorphisms.

Proposition 1 states that for $a > 0$, there is a branch of the unstable manifold of a hyperbolic fixed point which crosses the symmetry line $y = -x$. This is proven in Devaney [5]; however, our proof uses the curvature of the manifold to obtain this result more directly. As a result of Proposition 1 we can conclude the existence of a symmetric homoclinic point, $\mathcal{X}_c$. Theorem 1 then states that the crossing of C^u at $\mathcal{X}_c$ is transverse. A corollary to this is that there exists a horseshoe and that the first symmetric homoclinic point is transversal, as conjectured by Devaney.

1.1. *Short sketch of the proof.* We denote the measure-preserving, orientation-preserving Hénon map as H. The key to our proof is that if we have an oriented curve in the plane

$$\mathbf{z}(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix}$$

which has negative curvature and for which $\dot{z}_1(t) > 0$, then H transforms this curve to one that also has negative curvature.

Using this idea, we show that a branch of the unstable manifold, C^u, of a hyperbolic fixed point can be parametrized so that $\dot{z}_1(t) > 0$ for small t. We show that C^u initially has negative curvature and lies above the line $y = -x$, hereafter denoted as $\mathcal{L}$. We then follow C^u to the right until at some point, $\mathbf{z}(t_1)$, we must have $\dot{z}_1(t_1) = 0$. We show that $\mathrm{H}(\mathbf{z}(t_1))$ is below $\mathcal{L}$. It is elementary to show that the stable and unstable manifolds are symmetric about $\mathcal{L}$ by reflection. At this point we have established that for every value of $a > 0$ there is a point $\mathcal{X}_c$ where C^u crosses $\mathcal{L}$. By symmetry, a branch of the stable manifold must also cross $\mathcal{L}$ and at $\mathcal{X}_c$. Hence $\mathcal{X}_c$ is a homoclinic point. In Theorem 1 we show that $\mathcal{X}_c$ is a nondegenerate homoclinic point. Thus we have the corollary that for $a > 0$ H always has a horseshoe.

1.2. *Notation.* H is measure-preserving and orientation-preserving and so

$$\mathrm{H}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 + y - a x^2 \\ -x \end{pmatrix}.$$

There are two fixed points for H. The equations for the fixed points are

$$a x^2 + 2x - 1 = 0 \quad y = -x$$

and the two fixed points are:

$$\mathbf{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} -(1 + \sqrt{1+a})/a \\ (1 + \sqrt{1+a})/a \end{pmatrix}$$

$$\mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} (-1 + \sqrt{1+a})/a \\ (1 - \sqrt{1+a})/a \end{pmatrix}$$

$\mathbf{p}$ is to the left of the vertical axis and $\mathbf{q}$ is to the right. In a later section it is shown that $\mathbf{p}$ is a hyperbolic saddle.

$W^u(\mathbf{p})$ and $W^s(\mathbf{p})$ are the unstable and stable manifolds at $\mathbf{p}$. Since $\mathbf{p}$ is a hyperbolic saddle $W^u(\mathbf{p})$ has two branches. C^u is the branch, which is shown to exist later, that starts to the right of $\mathbf{p}$.

For a matrix A, A^T is the transpose of A.

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and $\langle \mathbf{a}, \mathbf{b} \rangle$ is the vector inner product of two vectors $\mathbf{a}, \mathbf{b}$.

We use the notation $\mathrm{fix}\{\Phi\}$ for the fixed points of a map Φ.

2. *Geometry of the Hénon map*

In this section we discuss the symmetries of H and the basic structure of $W^u(\mathbf{p})$. Refer to Figure 1.

2.1. *Symmetries of H.* Let

$$\mathrm{L}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 + x - a\,y^2 \\ y \end{pmatrix}$$

and let

$$\mathrm{R}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}$$

then

$$\mathrm{H} = \mathrm{L} \circ \mathbf{B}$$

and

$$\mathrm{H} = (\mathrm{LR}) \circ (\mathrm{R}\mathbf{B})$$

since $\mathrm{R}^2 = \mathbf{I}$. Further,

$$(\mathrm{LR})^2 = (\mathrm{R}\mathbf{B})^2 = \mathbf{I}$$

so that H is the composition of two involutions. From this we have

$$\mathrm{H}^{-1} = (\mathrm{R}\mathbf{B}) \circ (\mathrm{LR})$$

H is topologically conjugate to its inverse by each of these involutions since

$$\mathrm{R}\mathbf{B} \circ \mathrm{H} = (\mathrm{R}\mathbf{B}) \circ (\mathrm{LR} \circ \mathrm{R}\mathbf{B}) = (\mathrm{R}\mathbf{B} \circ \mathrm{LR}) \circ (\mathrm{R}\mathbf{B}) = \mathrm{H}^{-1}\mathrm{R}\mathbf{B}$$

and

$$\mathrm{H} \circ (\mathrm{LR}) = (\mathrm{LR} \circ \mathrm{R}\mathbf{B}) \circ (\mathrm{LR}) = (\mathrm{LR})\mathrm{H}^{-1}$$

Each of these involutions has a one-dimensional manifold of fixed points.

LEMMA 1.

$$\mathrm{fix}\{\mathrm{R}\mathbf{B}\}\} = \left\{\begin{pmatrix} x \\ y \end{pmatrix} | \mathrm{y} = -\mathrm{x}\right\}$$

$$\mathrm{fix}\{\mathrm{LR}\} = \left\{\begin{pmatrix} x \\ y \end{pmatrix} | \mathrm{x} = .5(1 - \mathrm{a}\,\mathrm{y}^2)\right\}$$

Proof. Direct computation. □

Let $\mathrm{fix}\{\mathrm{R}\mathbf{B}\} = \mathcal{L}$ and $\mathrm{fix}\{\mathrm{LR}\} = \mathcal{P}$. Geometrically, the map **RB** is the perpendicular reflection across $\mathcal{L}$. The map LR is a horizontal reflection across $\mathcal{P}$. The image of a point (x, y) is the point (x', y) with $x \neq x'$ at the same distance from $\mathcal{P}$.

LEMMA 2.

$$\mathrm{R}\mathbf{B}(\mathbf{p}) = \mathbf{p} = \mathrm{LR}(\mathbf{p})$$

Proof. Direct computation. □

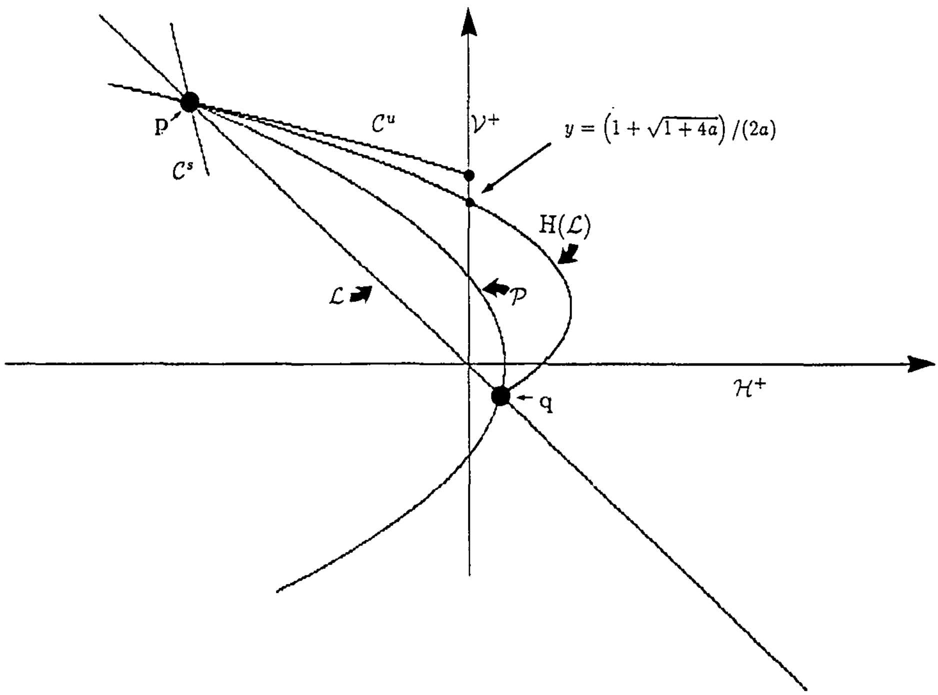

FIGURE 1.

The two fixed points of H, **p** and **q**, are the points of intersection of the curves $\mathcal{L}$ and $\mathcal{P}$.

The unstable manifold $W^u(\mathbf{p})$ of H at the point **p** is

$$W^u(\mathbf{p}) = \{\mathbf{z} \mid \mathrm{H}^{-n}(\mathbf{z}) \to \mathbf{p}\}$$

The stable manifold $W^s(\mathbf{p})$ of H at the point **p** is

$$W^s(\mathbf{p}) = \{\mathbf{z} \mid \mathrm{H}^{n}(\mathbf{z}) \to \mathbf{p}\}$$

We have the following lemmas:

LEMMA 3.

$$W^u(\mathbf{p}) = \mathrm{LR}(W^s(\mathbf{p})) = \mathrm{R}\mathbf{B}(W^s(\mathbf{p}))$$

Proof. Since $\mathrm{H}(W^u(\mathbf{p})) = W^u(\mathbf{p})$ and $\mathrm{H}(W^s(\mathbf{p})) = W^s(\mathbf{p})$ we need only prove $W^u(\mathbf{p}) = \mathrm{LR}(W^s(\mathbf{p}))$. We prove only one side of this equation,

$$W^u(\mathbf{p}) \subset \mathrm{LR}(W^s(\mathbf{p})).$$

The second half is analogous. Let $\mathbf{a} \in W^u(\mathbf{p})$, then $\mathrm{H}^{-n}(\mathbf{a}) \to \mathbf{p}$ and so

$$\mathrm{LR}(\mathrm{H}^{-n}(\mathbf{a})) \to \mathrm{LR}(\mathbf{p}) = \mathbf{p}.$$

By topological conjugacy and continuity

$$\mathrm{H}^n(\mathrm{LR}(\mathbf{a})) \to \mathbf{p}$$

so that $\mathrm{LR}(\mathbf{a}) \in W^s(\mathbf{p})$ or $\mathbf{a} \in \mathrm{LR}(W^s(\mathbf{p}))$. This gives $W^u(\mathbf{p}) \subset \mathrm{LR}(W^s(\mathbf{p}))$. □

LEMMA 4. *Let* $C^s = \mathbf{RB}(C^u)$, *then* C^s *is a branch of* $W^s(\mathbf{p})$.

Proof. Since $\mathbf{RB}(W^u(\mathbf{p})) = W^s(\mathbf{p})$ we have $\mathbf{RB}(C^u) \subset W^s(\mathbf{p})$. Since $\mathbf{p} \in C^u$, $\mathbf{p} \in \mathbf{RB}(C^u)$. Since the eigenvalues of $\mathrm{DH}(\mathbf{p})$ are positive each of the two branches of $W^u(\mathbf{p})$ or $W^s(\mathbf{p})$ are mapped onto itself by H. Thus $\mathbf{RB}(C^u)$ is the subset of a branch of $W^s(\mathbf{p})$. By topological conjugacy it must be the entire branch. □

2.2. *The basic structure of* $W^u(\mathbf{p})$. Let DH be the derivative of H.

LEMMA 5.

$$\mathrm{DH}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2a\,x & 1 \\ -1 & 0 \end{pmatrix},$$

and $\det(\mathrm{DH}) = 1$. *The trace of this matrix is given by* $\mathrm{tr}(\mathrm{DH}) = -2a\,x$.

Proof. Direct computation. □

For measure-preserving maps of the plane, the type of fixed point is determined by $(\mathrm{tr}/2)^2 - 1$, which is be called the discriminant, D . If D> 0, then the fixed point is hyperbolic. If D< 0 the point is elliptic; if D=0 equal to 0 it is called parabolic by some authors.

For $\mathbf{p}$, D= $1 + a + 2\sqrt{1+a}$ so that $\mathbf{p}$ is hyperbolic for all $a > 0$. When $a < 3$, $\mathbf{q}$ is elliptic (distinct complex conjugate eigenvalues on the unit circle). If $a = 3$, $\mathrm{DH}(\mathbf{q})$ has eigenvalues -1,-1. For $a > 3$, $\mathbf{q}$ is hyperbolic and $\mathrm{DH}(\mathbf{q})$ has negative eigenvalues.

The slope of the unstable manifold at $\mathbf{p}$ is provided by the following lemma, see Figure 1:

LEMMA 6. (1) *The expanding eigenvalue at* $\boldsymbol{p}$ *is*

$$\lambda_u = 1 + \sqrt{1+a} + \sqrt{1 + a + 2\sqrt{1+a}} > 2 + \sqrt{3} > 1$$

(2) *The slope,* s_1, *of* $W^u(\mathbf{p})$ *is*

$$s_1 = -1/\lambda_u$$

(3) $-1 < s_1 < 0$.

(4) *The contracting eigenvalue is*

$$\lambda_s = 1/\lambda_u.$$

(5) *The slope of* $W^s(\mathbf{p})$ *is* $-1/\lambda_s = -\lambda_u$

(6) *The slope of* $\mathcal{P}$ *at* $\boldsymbol{p}$ *is* $1/(ap_1)$

Proof. Direct computations. □

By (2) the slope of both $W^u(\mathbf{p})$ and $W^s(\mathbf{p})$ is negative at $\mathbf{p}$.

Because of its significance in the proof of our results we state the following fact as a separate lemma:

LEMMA 7. *The slope of $W^u(\mathbf{p})$ at p is greater than the slope of $\mathcal{L}$ and the slope of $\mathcal{P}$ at p.*

Proof. The relation between $\mathcal{L}$ and C^u follows from (1) and (2). The inequality of the slope of $\mathcal{P}$ and C^u follows from (6) and the fact that

$$\lambda_u = -a\,p_1 + \sqrt{1 + a + 2\sqrt{1+a}}.$$

□

This lemma says that as $W^u(\mathbf{p})$ advances to the right in a neighborhood of $\mathbf{p}$, it lies above $\mathcal{L}$ and $\mathcal{P}$. A key point in our proof will be to show that $W^u(\mathbf{p})$ must cross $\mathcal{L}$ again before crossing $\mathcal{P}$

From this lemma we know that there is a branch, C^u, of $W^u(\mathbf{p})$ that starts to the right of $\mathbf{p}$.

Let

$$\mathbf{z}(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix}$$

be a parameterization of of C^u such that $\mathbf{z}(0)$=$\mathbf{p}$, $\dot{\mathbf{z}}(t) \neq 0$ and $\dot{z}_1(t) > 0 > \dot{z}_2(t)$ for t in some interval $[0, \delta), \delta > 0$.

From this point onward $\mathbf{z}(t)$ always refers this parameterization of C^u.

LEMMA 8. *Let* $\mathrm{H}(\mathcal{L}) = \mathrm{H}_{\mathcal{L}}$. *$C^u$ cannot intersect* $\mathrm{H}_{\mathcal{L}}$ *before it intersects $\mathcal{L}$.*

Proof. Assume C^u intersects $\mathrm{H}_{\mathcal{L}}$ at a point $r \neq \mathbf{p}$ and has not intersected $\mathcal{L}$ and let $r = \mathrm{H}(s)$. Then s precedes r, $s \prec r$, and $\mathrm{H}^{-1}(r) = \mathrm{H}^{-1}(\mathrm{H}(s)) = s \in \mathcal{L}$ which gives a contradiction.

□

LEMMA 9. *Let $\mathcal{V}^+$ be the positive vertical axis.* $\mathrm{H}_{\mathcal{L}}$ *is a parabola which opens to the left, passes through $\mathbf{p}$, crosses $\mathcal{V}^+$ at $y = (1 + \sqrt{4a+1})/(2a)$, crosses the horizontal axis at $x = 1$, crosses $\mathcal{L}$ again at $\mathbf{q}$, crosses the negative vertical axis at $y = -(-1 + \sqrt{4a+1})/(2a)$. From $\mathbf{p}$ to $\mathbf{q}$,* $\mathrm{H}_{\mathcal{L}}$ *lies above $\mathcal{L}$.*

Proof. Direct computation. □

LEMMA 10. *Suppose $\dot{z}_1(t) > 0$ for all t. Then C^u cannot intersect* $\mathrm{H}_{\mathcal{L}}$ *before it intersects $\mathcal{V}^+$.*

Proof. If C^u intersects $\mathrm{H}_{\mathcal{L}}$ then it must have already crossed $\mathcal{L}$ by Lemma 8. Since $\dot{z}_1(t) > 0$, C^u cannot cross $\mathcal{L}$ until after it has crossed $\mathcal{V}^+$. Thus the conclusion follows.

□

3. *Geometry of* C^u

The curvature of an oriented plane curve, $\mathbf{w}(t)$, is

$$\kappa(t) = \kappa(\mathbf{w}(t)) = \frac{\langle \mathbf{B}^{-1}\dot{\mathbf{w}}, \ddot{\mathbf{w}} \rangle}{\|\dot{\mathbf{w}}\|^3} = \frac{\dot{w}_1(t)\ddot{w}_2(t) - \dot{w}_2(t)\ddot{w}_1(t)}{\|\dot{\mathbf{w}}\|^3}$$

Since the quantity $\|\dot{\mathbf{w}}\|$ is always positive we may omit it from consideration and obtain a formula for the sign of the curvature:

$$\mathrm{sgn}(\kappa(t)) = \mathrm{sgn}(\langle \mathbf{B}^{-1}\dot{\mathbf{w}}, \ddot{\mathbf{w}} \rangle)$$

3.1. *Sign of curvature.* We have the following lemma relating the sign of the curvature of a plane curve $\mathbf{z}(t)$ to the sign of the curvature of its image under H:

LEMMA 11. (Curvature Lemma.) *Let*

$$\mathbf{w}(t) = \mathrm{H}(\mathbf{z}(t)).$$

Then

$$\langle \mathbf{B}^{-1}\dot{\mathbf{w}}, \ddot{\mathbf{w}} \rangle = \langle \mathbf{B}^{-1}\dot{\mathbf{z}}, \ddot{\mathbf{z}} \rangle - 2a\dot{z}_1(t)^3$$

hence,

$$\mathrm{sgn}(\kappa(\mathbf{w}(t)) = \mathrm{sgn}(\langle \mathbf{B}^{-1}\dot{\mathbf{w}}, \ddot{\mathbf{w}} \rangle) = \mathrm{sgn}(\langle \mathbf{B}^{-1}\dot{\mathbf{z}}, \ddot{\mathbf{z}} \rangle - 2a\dot{z}_1(t)^3)$$

Proof. Direct computation. □

The curvature lemma says that if the curvature for an oriented plane curve $\mathbf{z}(t)$ is negative and $\dot{z}_1(t) > 0$ along this curve then the curvature for $\mathrm{H}(\mathbf{z}(t))$ is also negative. If the curvature of an oriented plane curve $\mathbf{z}(t)$ is positive and $\dot{z}_1(t) < 0$ along this curve then the curvature for $\mathrm{H}(\mathbf{z}(t))$ is also positive. Thus, the curvature lemma provides sufficient conditions for H, when considered as a mapping on a set of plane curves, to preserve the sign of curvature.

3.2. *Curvature of* C^u *at* $\mathbf{p}$. The curvature of C^u at $\mathbf{p}$ is given in the following lemma:

LEMMA 12. (Initial curvature.)

$$\kappa(0) = \frac{-2a\,\lambda_u^3}{(\lambda_u^2 + 1)^{3/2}(\lambda_u^3 - 1)} < 0$$

Proof. By Lemma 11 we have

$$\langle \mathbf{B}^{-1}\dot{\mathbf{w}}, \ddot{\mathbf{w}} \rangle = \langle \mathbf{B}^{-1}\dot{\mathbf{z}}, \ddot{\mathbf{z}} \rangle - 2a(\dot{z}_1(t))^3$$

Dividing both sides by $\|\dot{\mathbf{w}}\|^3$, simplifying, and omitting t:

$$\kappa(\mathbf{w}) = \kappa(\mathbf{z})\frac{\|\dot{\mathbf{z}}\|^3}{\|\dot{\mathbf{w}}\|^3} - 2a\frac{\dot{z}_1^3}{\|\dot{\mathbf{w}}\|^3}.$$

Evaluating this expression at $t = 0$ we have

$$\kappa(0) = \kappa(0)\left(\frac{\|\dot{\mathbf{z}}(0)\|}{\|\dot{\mathbf{w}}(0)\|}\right)^3 - 2a\left(\frac{\dot{z}_1(0)}{\|\dot{\mathbf{w}}(0)\|}\right)^3$$

Now

$$\frac{\|\dot{\mathbf{z}}(0)\|^3}{\|\dot{\mathbf{w}}(0)\|^3} = 1/\lambda_u^3,$$

and

$$\frac{\dot{z}_1^3(0)}{\|\dot{\mathbf{w}}(0)\|^3} = 1/(\lambda_u^2 + 1)^{3/2}.$$

Simplifying and solving for $\kappa(0)$ we obtain the formula for curvature. Since $\lambda_u > 1$ the curvature is negative. □

3.3. *Orientation along C^u.* We say that one point $\mathbf{z}(t_1)$ *precedes another point* $\mathbf{z}(t_2)$ on $\mathbf{z}(t)$ when the arc length to $\mathbf{z}(t_1)$ is shorter than the arc length to $\mathbf{z}(t_2)$ and we write this as $\mathbf{z}(t_1) \prec \mathbf{z}(t_2)$. We say a point, $\mathbf{z}(t_\eta)$, is between two points $\mathbf{z}(t_1) \prec \mathbf{z}(t_2)$ when $\mathbf{z}(t_1) \prec \mathbf{z}(t_\eta) \prec \mathbf{z}(t_2)$. We have the following facts about H and $\mathbf{z}(t)$.

LEMMA 13. (1) *Given t_1 let t_2 be such that*

$$\mathbf{z}(t_2) = \mathrm{H}(\mathbf{z}(t_1))$$

then $t_2 > t_1$.

(2) *If $\mathbf{z}(t_*) \prec \mathbf{z}(t_2) \prec \mathrm{H}(\mathbf{z}(t_*))$ then there exist t_1 such that $\mathbf{z}(t_2) = \mathrm{H}(\mathbf{z}(t_1))$ and $t_1 < t_*$.*

Proof. Both items follow from the fact that H has only one fixed point in C, H is 1–1, onto, and H expands C^u near $\mathbf{p}$. □

4. *Turning lemmas*

In this section we present several lemmas that tell us when C^uchanges direction.

LEMMA 14. *Let*

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

be any vector in $\mathbf{R}^2$ with $\alpha > 0$ and $\beta < 0$. Let

$$\mathbf{z} = \begin{pmatrix} x \\ y \end{pmatrix}$$

be any point in $\mathbf{R}^2$ with $x > 0$. Then the vector

$$\mathrm{DH}(\mathbf{z})\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

has positive slope.

Proof.

$$\mathrm{DH}(\mathbf{z})\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = -\begin{pmatrix}2a\,x\alpha-\beta\\ \alpha\end{pmatrix}.$$

Since $a > 0, x > 0, \alpha > 0$, and $\beta < 0$ the slope is positive. □

LEMMA 15. *There exists a point on* $\mathbf{z}(t)$ *where* $\dot{z}_1(t) = 0$.

Proof. Assume $\dot{z}_1(t) > 0$ for all t. $\mathcal{C}^u$ is bounded below by $\mathrm{H}_{\mathcal{L}}$until after it crosses $\mathcal{V}^+$by Lemma 10. By Lemma 11 $\mathcal{C}^u$ must continue to the right and downward as long as $\dot{z}_1(t) > 0$. Hence either $\mathbf{z}(t)$ crosses $\mathcal{V}^+$ or it stops at a fixed point $\mathbf{q}$. But to reach $\mathbf{q}$, $\mathbf{z}(t)$ must cross $\mathcal{V}^+$. Hence, in any case, $\mathcal{C}^u$ must cross $\mathcal{V}^+$. Further, $\mathcal{C}^u$ crosses $\mathcal{V}^+$ with negative slope so that we may apply Lemma 14 and continuity to obtain a point with vertical slope. □

Let $\mathbf{z}(t_1)$ be the first point where $\dot{z}_1(t) = 0$.

LEMMA 16. $\dot{z}_2(t_1) \neq 0$, *and* $\dot{z}_2(t_1) < 0$.

Proof. Suppose that $\dot{z}_2(t_1) = 0$. Applying H^{-1} to $\mathbf{z}(t_1)$ we obtain an earlier point where $\dot{z}_1(t) = 0$. Hence, $\dot{z}_1(t_1) = 0$ and $\dot{z}_2(t_1) \neq 0$. Since $\dot{z}_2(t) < 0$ for $t < t_1$, and $\dot{z}_2(t_1) \neq 0$, by continuity we have $\dot{z}_2(t_1) < 0$. □

LEMMA 17. *Let* $\mathbf{z}(t_2) = \mathrm{H}(\mathbf{z}(t_1))$. *Then* $\mathbf{z}(t_2)$ *is the first point were* $\dot{z}_2(t_2) = 0$.

Proof. Suppose there exists a time $t_1 < t_\psi < t_2$ where $\dot{z}_2(t_\psi) = 0$. We have $\mathbf{z}(t_1) < \mathbf{z}(t_\psi) < \mathbf{z}(t_2) = \mathrm{H}(\mathbf{z}(t_1))$. By Lemma ii, there exists a $t_\eta < t_1$ such that $\mathbf{z}(t_\psi) = \mathrm{H}(\mathbf{z}(t_\eta))$. Then

$$\begin{aligned}0 &= \dot{z}_2(t_\psi) = \langle \mathrm{DH}(\mathbf{z}(t_\eta))\dot{\mathbf{z}}(t_\eta), \mathbf{e}_2\rangle = \langle \dot{\mathbf{z}}(t_\eta), \mathrm{DH}^T(\mathbf{z}(t_\eta))\mathbf{e}_2\rangle\\ &= -\langle \dot{\mathbf{z}}(t_\eta), \mathbf{e}_1\rangle = 0\end{aligned}$$

Thus t_1 was not the first occurrence where $\langle \dot{\mathbf{z}}(t), \mathbf{e}_1\rangle = 0$. □

5. *Symmetric homoclinic points*

LEMMA 18. *Let* $\mathbf{z}(t), t \in [0, 1]$ *be a* C^2 *curve in* $\mathbf{R}^2$. *Assume*

(1) $\dot{z}_1(0) = 0$ *and* $\dot{z}_2(1) = 0$
(2) $\dot{z}_2(t) < 0$ *for* $0 \leq t < 1$
(3) $\dot{z}_1(t)\ddot{z}_2(t) - \dot{z}_2(t)\ddot{z}_1(t) < 0$ *for all* t.
(4) *Then* $\dot{z}_1(t) < 0$ *for* $0 < t < 1$ *and* $z_1(1) < z_1(0)$

Proof. Let $r(t) = \dot{z}_1(t)/\dot{z}_2(t)$ for $0 \leq t < 1$. Then $r(0) = 0$ by (1) and $\dot{r}(t) > 0$ by (3). Hence $r(t)$ is strictly increasing and therefore must be positive. Since $\dot{z}_2(t) < 0$ by (2) we have $\dot{z}_1(t) < 0$ for $0 < t < 1$ and so $\dot{z}_1(1) < \dot{z}_1(0)$. □

LEMMA 19. *Let* $\mathbf{u}(t)$ *be the segment of* $\mathcal{C}^u$ *that starts at* $\mathbf{z}(t_1)$ *and ends at* $\mathrm{H}(\mathbf{z}(t_1)) = \mathbf{z}(t_2)$, *then* $\mathbf{u}(t)$ *satisfies the hypothesis of Lemma 18.*

Proof. From our hypothesis about $\mathbf{z}(t_1)$ we have $\dot{u}_1(t_1) = 0$ and $\dot{u}_2(t_2) = 0$. Since this is the first point where $\dot{z}_1 = 0$ we must have $\dot{u}_2(t) < 0$ for all points of $\mathbf{u}(t)$ by Lemmas 16 and 17. The third condition of Lemma 18 is the condition of negative curvature which follows from Lemma 11. □

LEMMA 20. *Let* $\mathbf{z}(t_2) = \mathrm{H}(\mathbf{z}(t_1))$, *then* $\mathbf{z}(t_2)$ *lies below* $\mathcal{L}$.

Proof. By Lemma 19 we know that we must have $z_1(t_2) < z_1(t_1)$. But $z_2(t_2) = -z_1(t_1)$ so that $z_1(t_2) < -z_2(t_2)$, or equivalently, $z_2(t_2) < -z_1(t_2)$ and so $\mathbf{z}(t_2)$ is below $\mathcal{L}$. □

Figure 2 illustrates the following two results.

LEMMA 21. *The curvature of* C^u *is negative before crossing* $\mathcal{L}$.

Proof. The curvature of C^u is initially negative for small t by Lemma 12 and continuity. The curvature of C^umust continue negative until $\mathbf{z}(t_2) = \mathrm{H}(\mathbf{z}(t_1))$ (using the notation of Lemma 20) by Lemmas 11 and 20. But at $\mathbf{z}(t_2)$ C^u has crossed $\mathcal{L}$. □

PROPOSITION 1. *For any positive value of the parameter a,* C^u *crosses* $\mathcal{L}$.

Proof. From Lemma 6 we can conclude that C^u is above $\mathcal{L}$ for small $t > 0$. By Lemma 15 there exist a first point, $\mathbf{z}(t_1)$, on $C^u$where $\dot{z}_1(t_1) = 0$. By Lemma 20 $\mathrm{H}(\mathbf{z}(t_1))$ is below $\mathcal{L}$. Hence C^u has crossed $\mathcal{L}$. □

Let $\mathcal{X}_c = (x_c, -x_c)$ be the first point where C^u crosses $\mathcal{L}$.

LEMMA 22. C^u *and* C^s*must cross* $\mathcal{L}$ *at the same point.*

Proof. This follows from the symmetry relations for C^u and C^s. □

6. *Crossings of* C^u *with* $\mathcal{V}^+$, $\mathcal{H}^+$ *and* $\mathcal{L}$

Refer to Figure 2 for an illustration of the ideas of this section. Let $\mathcal{L}_0$ be the segment of $\mathcal{L}$ from $\mathbf{p}$ to the origin, (0,0), let $C^u_{y_+}$be the arc of C^u which starts at $\mathbf{p}$ and ends at the first crossing of C^u with $\mathcal{V}^+$, and let $C^u_{x_+}$be the arc of C^u which starts at $\mathbf{p}$ and ends at the first crossing of $C^u$with $\mathcal{H}^+$, and let C^u_cbe the arc of C^u that starts at $\mathbf{p}$and ends at $\mathcal{X}_c$. Let $C^u_{y_+} \cap \mathcal{V}^+ = \{(0, y_+)\}$ and $C^u_{x_+} \cap \mathcal{H}^+ = \{(x_+, 0)\}$.

LEMMA 23.

$$\mathrm{H}(C^u_{y_+}) = C^u_{x_+}$$

Proof. Direct computation. □

LEMMA 24. *Let* $\mathbf{g} = (0, 1/a)$ *and let* $\mathcal{L}_{\mathrm{g}}$ *be the line from* $\mathbf{p}$ *to g, then* $C^u_{y_+} \cap \mathcal{L}_{\mathrm{g}} = \{\mathbf{p}\}$ *and therefore* $\mathrm{H}(\mathcal{L}_{\mathrm{g}}) \cap C^u_{x_+} = \{\mathbf{p}\}$

Proof. $C^u_{y_+}$is bounded away from $\mathcal{L}_{\mathrm{g}}$ by $\mathrm{H}_{\mathcal{L}}$. □

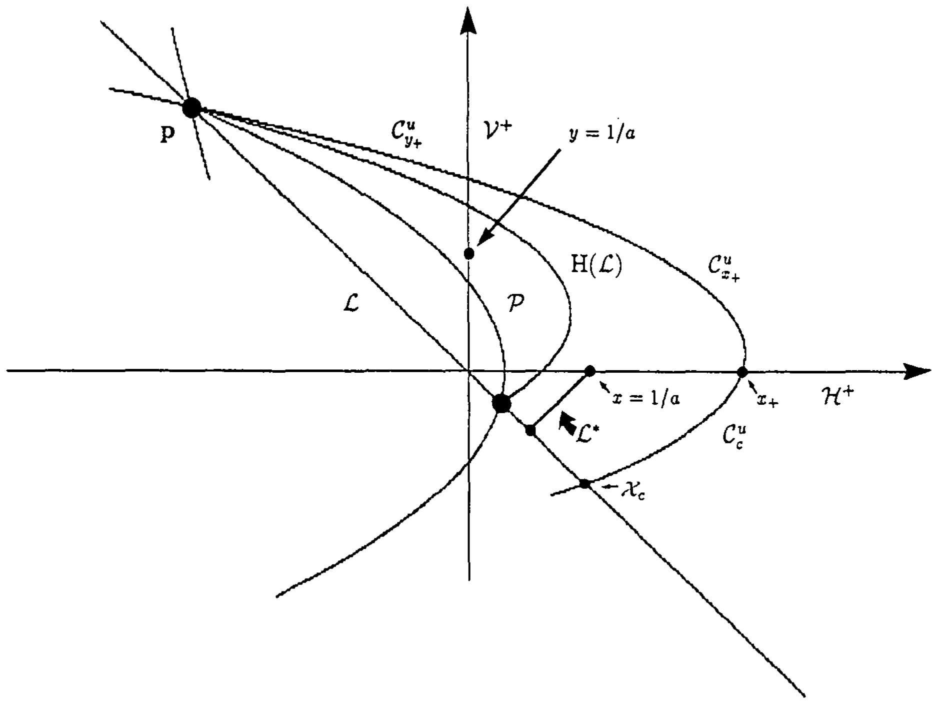

FIGURE 2.

LEMMA 25.

$$1/a < x_+$$

Proof.

$$\mathrm{H}(\mathbf{g}) = \begin{pmatrix} 1 + 1/a \\ 0 \end{pmatrix}$$

and $C^u_{x_+}$ must cross $\mathcal{H}^+$ above H(**g**) by the previous lemma, and hence above $\mathbf{h} = (1/a, 0)$. We conclude that $1/a < x_+$. □

Let $m(t) = \dot{z}_2(t)/\dot{z}_1(t) = 1/r(t)$ be the slope of C^u at a point t. Note that $\mathbf{z}(t_c) = \mathcal{X}_c$.

LEMMA 26. *Let* $\mathrm{H}^{-1}(\mathbf{z}(t_c)) = \mathbf{z}(t_{-1})$. *Then* $m(t_{-1}) < 0$.

Proof. (1) Assume that $\mathbf{z}(t_c) \prec \mathbf{z}(t_1)$ then the result follows by the definition of $\mathbf{z}(t_1)$. (2) Assume $\mathbf{z}(t_1) \prec \mathbf{z}(t_c)$ then since $\mathbf{z}(t_c) \prec \mathbf{z}(t_2) = \mathrm{H}(\mathbf{z}(t_1))$ by Lemma 20 and so $\mathrm{H}^{-1}(\mathbf{z}(t_c)) \prec \mathbf{z}(t_1)$ by Lemma 13 and so $m(t_{-1}) < 0$. □

LEMMA 27. *Let* $m_c = m(t_c)$, *then for* $t_1 < t \le t_c$ *we have* $m(t) \ge m_c$.

Proof. Lemma 21. □

LEMMA 28. *Let C_1 be the curve defined by $y = f(x) = mx + b$ and let C_2 be the curve defined by $y = g(x)$ where $g(0) > b$ and $g'(x) \geq m$, then $C_1 \cap C_2 = \emptyset$.*

Proof. If not, there is an x_1 such that $g(x_1) = mx_1 + b$. from this we conclude that

$$m \leq \frac{g(x_1) - g(0)}{x_1} = \frac{g(x_1) - b}{x_1} - \frac{g(0) - b}{x_1} = m - \frac{g(0) - b}{x_1} < m$$

a contradiction. □

LEMMA 29. *Let $\mathcal{L}^*$ the segment of the line $y = m_c(x - (1/a))$ from $\mathbf{h}$ to $\mathcal{L}$. Then $C_c^u \cap \mathcal{L}^* = \emptyset$.*

Proof. Rotate the plane 90 degrees and apply preceding lemma. □

Let $\mathbf{k} = (k_c, -k_c)$ be the point of intersection of $\mathcal{L}$ and $\mathcal{L}^*$. The dependence of $\mathbf{k}$ on m_c will not be indicated in the notation.

7. *Degenerate homoclinic points*

A degenerate homoclinic point is a point where $W^u(\mathbf{p})$ crosses $W^s(\mathbf{p})$ topologically but not transversely. We will assume in the following three lemmas that $\mathcal{X}_c$ is a degenerate homoclinic point.

LEMMA 30. $m(t_c) = 1$

Proof. Let

$$\dot{\mathbf{z}}(t_c) = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

then

$$\mathbf{RB}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

is the slope of C^sat $\mathcal{X}_c$. These two vectors point in the opposite direction therefore

$$\mathbf{RB}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = -\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

when $\mathcal{X}_c$ is degenerate and hence $\alpha = \beta$. □

LEMMA 31.

$$\mathrm{DH}^{-1}(\mathbf{z}(t_c))\begin{pmatrix} \alpha \\ \alpha \end{pmatrix} = \alpha\begin{pmatrix} -1 \\ 1 - 2ax_c \end{pmatrix}.$$

Proof. Direct computation. □

LEMMA 32. *$x_c > 1/2a$ and so $2ax_c - 1 > 0$.*

Proof. If $\mathcal{X}_c$ is degenerate then $m(t_c) = m_c = 1$, $1/a < x_+$, and $\mathbf{z}(t_1) \prec \mathcal{X}_c$. By Lemma 29 $k_c < x_c$. By a direct computation $k_c = 1/2a$. □

8. *Embedded horseshoes in H*

THEOREM 1. H *has a transverse homoclinic point.*

Proof. $\mathcal{X}_c$ is a homoclinic point. Suppose that it is not transverse. By Lemma 30 $m(t_c) = 1$ so that

$$\dot{\mathbf{z}}(t_c) = \begin{pmatrix} \alpha \\ \alpha \end{pmatrix}$$

for some α. By Lemma 31 $\mathrm{DH}^{-1}(\mathcal{X}_c)(\dot{\mathbf{z}}(t_c)) = \begin{pmatrix} -1 \\ 1 - 2ax_c \end{pmatrix}$. Hence by Lemma 32 $m(t_{-1}) > 0$. But $\mathrm{H}^{-1}(\mathcal{X}_c) \prec \mathbf{z}(t_1)$ and must have negative slope by Lemma 26, a contradiction. We conclude that the crossing of C^u and C^sat $\mathcal{X}_c$ is transverse. □

COROLLARY.H *has an embedded horseshoe.*

Proof. Smale [**6**]. □

9. *Twist-and-flip map*

The twist-and-flip map, Brown [**2**], is

$$\mathrm{FT}\begin{pmatrix} x \\ y \end{pmatrix} = -\begin{pmatrix} (x-a)\cos(r) - y\sin(r) + a \\ (x-a)\sin(r) + y\cos(r) \end{pmatrix}$$

where $a > 0$. The technique used for the Hénon map can be carried over to the twist-and-flip map greatly improving both the proofs and the results of that paper. We state only two results.

LEMMA 33. (Twist-and-flip curvature.) *Let*

$$\mathbf{w}(t) = \mathrm{FT}(\mathbf{z}(t))$$

and $\lambda = r = \sqrt{(x-a)^2 + y^2}$ *then,*

(1)

$$\langle \mathbf{B}^{-1}\dot{\mathbf{w}}, \ddot{\mathbf{w}} \rangle = \langle \mathbf{B}^{-1}\dot{\mathbf{z}}, \ddot{\mathbf{z}} \rangle + \dot{r}\,\{3|\dot{\mathbf{z}}|^2 - \dot{r}^2 + (r\,\dot{r})^2 - 3\dot{r}\langle \mathbf{B}^{-1}(\mathbf{z}-\mathbf{a}), \dot{\mathbf{z}} \rangle\}$$

(2)

$$3|\dot{\mathbf{z}}|^2 - \dot{r}^2 + (r\,\dot{r})^2 - 3\dot{r}\langle \mathbf{B}^{-1}(\mathbf{z}-\mathbf{a}), \dot{\mathbf{z}} \rangle > 0$$

Proof. The proof is a direct computation. □

THEOREM. *If* $a > \pi/2$, *FT has a horseshoe.*

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