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Infinitesimal Stretching and Folding II: The Simple Scroll

Ray Brown

The EEASI Corporation
Houston, Texas, 77057 USA

Abstract

In [1] Infinitesimal Stretching and Folding (ISF) is explained and it is shown that ISF diffeomorphisms (ISFD) can duplicate the complex dynamics of ODEs with far greater computational efficiency than is possible using standard numerical methods. In this paper I examine the dynamics of a very simple ISFD, the Simple Scroll. Three areas will be examined: (1) trajectories; (2) Fixed points; and, (3) Poincaré maps.

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AMS (MOS) subject classification: 37D45.

1 Introduction

In [1] it is shown that ISF diffeomorphisms (ISFD) can duplicate the complex dynamics of ODEs with far greater computational efficiency than using conventional numerical methods to solve ODEs. In this paper, I introduce the simple scroll ISFD and examine its range of dynamics: trajectories, fixed points and Poincaré maps.

The value of ISFDs is that they are diffeomorphisms that have properties like flows while being discrete mappings of a manifold. Instead of only producing discrete values such as is common for diffeomorphisms such as the Twist-and-flip, Hénon, Chirikov, etc, the iterates of an ISFD look like trajectories of ODEs while having the form of a diffeomorphism.

2 Derivation of the Simple Scroll ISFD

The Chua double scroll will be the starting point to derive a very simple ISFD with scroll dynamics. The dimensionless form of Chua's equations are given by:

$$\dot{x} = \alpha(y - x - f(x))$$

$$\begin{aligned} \dot{y} &= x - y - z \\ \dot{z} &= -\beta y \end{aligned} \quad (1)$$

where,

$$f(x) = \begin{cases} bx + a - b & \text{for} \quad x \geq 1.0 \\ ax & \text{for} \quad |x| \leq 1.0 \\ bx - a + b & \text{for} \quad x \leq 1.0 \end{cases}$$

is a three segment piecewise linear function and α, β are dimensionless parameters. In place of the classical Chua ODE, we will use a variant discussed in [2]. The formulation for Figure 1 is as follows:

$$\begin{aligned} u &= x + h \cdot (17 \cdot y - 3 \cdot (x - a \cdot f(x))) \\ v &= y + h \cdot ((x - a \cdot f(x)) - 3 \cdot y + z) \\ w &= z - h \cdot \beta \cdot y \end{aligned}$$

where

$$f(x) = 2 \cdot (\exp(\beta \cdot x) - 1)/(\exp(\beta \cdot x) + 1)$$

Figure 1

This figure is a modified version of the Chua Double Scroll from [2], $\beta = 12.6$, $a = 2.5$

Following [1] we replace $f(x)$ by a constant and derive the following equation as a starting point:

$$\begin{aligned} x &= \exp(\alpha \cdot t) \cdot ((x_0 - a) \cdot \cos(\omega \cdot t) + (y_0 - b) \cdot \sin(\omega \cdot t)) + a \\ y &= \exp(\alpha \cdot t) \cdot ((y_0 - b) \cdot \cos(\omega \cdot t) - (x_0 - a) \cdot \sin(\omega \cdot t)) + b \\ z &= \exp(\beta \cdot t) \cdot (z_0 - c) + c \end{aligned}$$

3 The Simple Scroll

Replace t by an infinitesimal step size h to get

$$\begin{aligned} x &= \exp(\alpha \cdot h) \cdot ((x_0 - a) \cdot \cos(\omega \cdot h) + (y_0 - b) \cdot \sin(\omega \cdot h)) + a \\ y &= \exp(\alpha \cdot h) \cdot ((y_0 - b) \cdot \cos(\omega \cdot h) - (x_0 - a) \cdot \sin(\omega \cdot h)) + b \\ z &= \exp(\beta \cdot h) \cdot (z_0 - c) + c \end{aligned}$$

To form a scroll, the center (a, b, c) must change with time as well as the dissipation in z. Since the time variable has been removed, an analog of the time varying term must be replaced by a function of the autonomous variables. It is known that using the sgn function will shift the center coordinates (a, b, c) to $(-a, -b, -c)$. But using the sgn function will insert an undesirable discontinuity into the equation. However, since we are working in three dimensions this can be avoided by using the first two terms of the Fourier expansion of the square wave function:

$$f(u) = \sin(u) + \sin(3 \cdot u)/3$$

Since dividing the second term by 3 does not contribute significantly to the discussion we will simplify f as follows:

$$f(u) = \sin(u) + \sin(3 \cdot u)$$

f will be called the transition function; when inserted into the infinitesimal equation we arrive at the ISFD:

$$\begin{aligned}
x_{n+1} &= \exp(\alpha \cdot h) \cdot ((x_n - a \cdot f(u)) \cdot \cos(\omega \cdot h) + (y_n - b \cdot f(u)) \cdot \sin(\omega \cdot h)) + a \cdot f(u) \\
y_{n+1} &= \exp(\alpha \cdot h) \cdot ((y_n - b \cdot f(u)) \cdot \cos(\omega \cdot h) - (x_n - a \cdot f(u)) \cdot \sin(\omega \cdot h)) + b \cdot f(u) \\
z_{n+1} &= \exp(\beta \cdot f(u) \cdot h) \cdot (z_n - c) + c
\end{aligned}$$

In the interest of further simplification I omit the parameters a, b and remove f from the second equation to get the Simple Scroll equation:

$$\begin{aligned}
x_{n+1} &= \exp(\alpha \cdot h) \cdot ((x_n - f(u)) \cdot \cos(\omega \cdot h) + y_n \cdot \sin(\omega \cdot h)) + f(u) \\
y_{n+1} &= \exp(\alpha \cdot h) \cdot (y_n \cdot \cos(\omega \cdot h) - (x_n - f(u)) \cdot \sin(\omega \cdot h)) \\
z_{n+1} &= \exp(\beta \cdot f(u) \cdot h) \cdot (z_n - c) + c
\end{aligned}$$

The choice of u is still to be determined. Note that f affects the damping in the z dimension in synchrony with change in the center of the x, y components of the trajectory.

The choice of u is best confined to a simple three-dimensional plane. By some trial-and-error we arrive at $u = x + z$. This completes the definition of the Simple Scroll.

The final form of the Simple Scroll ISFD is as follows:

$$\begin{aligned}
x_{n+1} &= \exp(\alpha \cdot h) \cdot ((x_n - f(u_n)) \cdot \cos(\omega \cdot h) + y_n \cdot \sin(\omega \cdot h)) + f(u_n) \\
y_{n+1} &= \exp(\alpha \cdot h) \cdot (y_n \cdot \cos(\omega \cdot h) - (x_n - f(u_n)) \cdot \sin(\omega \cdot h)) \\
z_{n+1} &= \exp(\beta \cdot f(u_n) \cdot h) \cdot (z_n - c) + c
\end{aligned}$$

where,

$$\begin{aligned}
u &= x + z \\
f(u) &= \sin(u) + \sin(3 \cdot u) \\
f'(u) &= \cos(u) + 3\cos(3 \cdot u) \\
f_x &= f_z \\
f_y &= 0
\end{aligned}$$

The Jacobian of this map is as follows:

$$\begin{pmatrix} a_{11} & \exp(\alpha \cdot h)\sin(\omega \cdot h) & a_{13} \\ a_{21} & \exp(\alpha \cdot h)\cos(\omega \cdot h) & \exp(\alpha \cdot h) f_z \\ a_{31} & 0 & a_{33} \end{pmatrix} \tag{2}$$

where,

$$\begin{aligned} a_{11} &= \exp(\alpha \cdot h)((1-f_x)\cdot\cos(\omega \cdot h)) + f_x \\ a_{13} &= \exp(\alpha \cdot h)(-f_z \cdot \cos(\omega \cdot h) + f_z \\ a_{21} &= -\exp(\alpha \cdot h)(1-f_x)\sin(\omega \cdot h) \\ a_{31} &= \exp(\beta \cdot f(u)\cdot h)\cdot \beta \cdot h \cdot f_x \cdot (z-1) \\ a_{33} &= \exp(\beta \cdot f(u)\cdot h)\cdot \beta \cdot h \cdot f_x \cdot (z-1) + \exp(\beta \cdot f(u) \cdot h) \end{aligned}$$

]

4 Trajectories of the Simple Scroll

The following three figures illustrate the complex dynamics of the Simple Scroll ISFD. Figure 2 presents three trajectories of the Simple Scroll ISFD with three different initial conditions. Parameter data is the same for all trajectories: $c = 1,\ \alpha = -0.02,\ \omega = 0.5,\ \beta = 0.007,\ h = 0.001$

[Figure 2]
Three Trajectories of the Simple Scroll ISFD with initial conditions, from top to bottom, $(1.52, 0, 1),\ (-3, 0, -\pi),\ (-3, 0, -2\pi)$.

Figure 3 illustrates a trajectory that originates at the unstable fixed point at the origin and terminates at a stable fixed point on the $z-$axis, $(0, 0, \pi/2)$.

[Figure 3]
Heteroclinic trajectory from the origin $(0, 0, 0)$ to $(0, 0, \pi/2)$.

Figure 4 illustrates a periodic limit cycle containing three scrolls. $c = 1,\ \alpha = -0.02,\ \omega = 0.5,\ \beta = 0.007,\ h = 0.001$ The transition function has been changed to $\sin(u) - \sin(3 \cdot u)$ and $u = x - z$

[Figure 4]
A periodic limit cycle containing three scrolls

Figure 5 is a variation of the Simple Scroll obtained by inserting a twist factor as used in the Twist-and-Flip map [3] in the rotational components.

$$\begin{aligned} x_n &= \exp(\alpha \cdot h)\cdot((x_n - f(u_n))\cdot\cos(\omega \cdot r \cdot h) + y_n \cdot \sin(\omega \cdot r \cdot h)) + f(u_n) \\ y_n &= \exp(\alpha \cdot h)\cdot(y_n \cdot \cos(\omega \cdot r \cdot h) - (x_n - f(u_n))\cdot \sin(\omega \cdot r \cdot h)) \\ z_n &= \exp(\beta \cdot f(u_n)\cdot h)\cdot(z_n - c) + c \end{aligned}$$

The transition function is $f(u)$, where, $u_1 = \cos((x+z) - 2\cdot(x+z)^2 + (x+z)^3)$
$u = (\sin(u_1) + \sin(3\cdot u_1)/3)\cdot\cos(u_1)^2$

Figure 5
The spiderweb scroll obtained by inserting a twist factor into the rotational components of the Simple Scroll
$\alpha = -0.02,\ \ \omega = 0.5,\ \ \beta = 0.007,\ \ r = \sqrt{x^2+y^2+z^3},\ \ h = 0.001$

5 Fixed Points of the Simple Scroll

At a fixed point we have

$$\begin{aligned} x_n &= \exp(\alpha\cdot h)\cdot((x_n - f(u_n))\cdot\cos(\omega\cdot h) + y_n\cdot\sin(\omega\cdot h)) + f(u_n) \\ y_n &= \exp(\alpha\cdot h)\cdot(y_n\cdot\cos(\omega\cdot h) - (x_n - f(u_n))\cdot\sin(\omega\cdot h)) \\ z_n &= \exp(\beta\cdot f(u_n)\cdot h)\cdot(z_n - c) + c \end{aligned}$$

Using the z equation, assuming $z_n \neq c$ we conclude that $f(u) = 0$, or $\sin(u) + \sin(3\cdot u) = 0$. An obvious fixed point is $x = y = z = 0$. Also, we conclude that $x = y = 0$ and $z = 2\cdot n\cdot\pi$ is a family of fixed points long the $z-$axis. Also, $u = (2\cdot n - 1)\pi/2$ provides an additional family of fixed points. It is routine to check that all but a select few of these fixed points are stable spirals. And the others are heteroclinic to one of the stable spirals. There are additional periodic points present as well.

6 Duffing Dynamics from the Simple Scroll

The classical Duffing Equation is

$$\ddot{y} + 0.05\dot{y} + y^3 = 7.5\cos(t) \tag{3}$$

The Duffing equation can be formulated as a three-dimensional autonomous system by the following rearrangement:

$$\begin{aligned} \dot{x} &= y & (4) \\ \dot{y} &= -0.05y - y^3 + 7.5\cos(z) & (5) \\ \dot{z} &= 1 & (6) \end{aligned}$$

The challenge is to replace $\dot{z} = 1$ with $\dot{z} = f(x,y,z)$ to remove any trace of the connection of Duffing Scroll to a time-varying forcing function. To produce a Duffing Scroll trajectory the following modifications to the simple scroll are made:

$$a = 0.95,\ \ c = 1,\ \ \alpha = -0.02,\ \ \omega = \sqrt{2}/8,\ \ \beta = 0.005,\ \ \gamma = \sqrt{3}/2,\ \ h = 0.001$$

The transition function is chosen as

$$f(u) = 1.5\cdot\cos(\gamma\cdot u) - \sin(u) \qquad g(u) = f(u-1)$$

and the transition plane is $u = x - z$. The initial conditions are $(0,0,2)$

Note that this is no longer a "simple scroll" in that the damping term in the $z-$coordinate does not match the the center of rotation term in the rotational component.

$$
\begin{aligned}
x_{n+1} &= \exp(\alpha \cdot h) \cdot ((x_n - a \cdot f(u)) \cdot \cos(\omega \cdot h) + y_n \cdot \sin(\omega \cdot h)) + a \cdot f(u) \\
y_{n+1} &= \exp(\alpha \cdot h) \cdot (y_n \cdot \cos(\omega \cdot h) - (x_n - a \cdot f(u)) \cdot \sin(\omega \cdot h)) \\
z_{n+1} &= \exp(g(u) \cdot \beta \cdot h) \cdot (z_n - c) + c
\end{aligned}
$$

The x-coordinate of the scroll is multiplied by 0.75 and the y-coordinate of the scroll is multiplied by 1.5 to bring out the similarities in the two trajectories, see Figure 6.

[Figure 6]
Trajectory A is the scaled Scroll; B is the Duffing trajectory; and, C is the overlay of the two trajectories

7 Poincaré Maps

The ISF formulation suggests a natural surface to be used for examine a Poincaré map or first-return map. In particular, for the Simple Scroll it is the function $f(u)$. Plotting the points where $f(u) = 0$ will produce an image of an attractor on that surface.

8 References

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[3] Brown, R. Chua,L. [1991] Horseshoes in the Twist and Flip Map *Int. J.Bifurcation and Chaos*, I(1), 235-252

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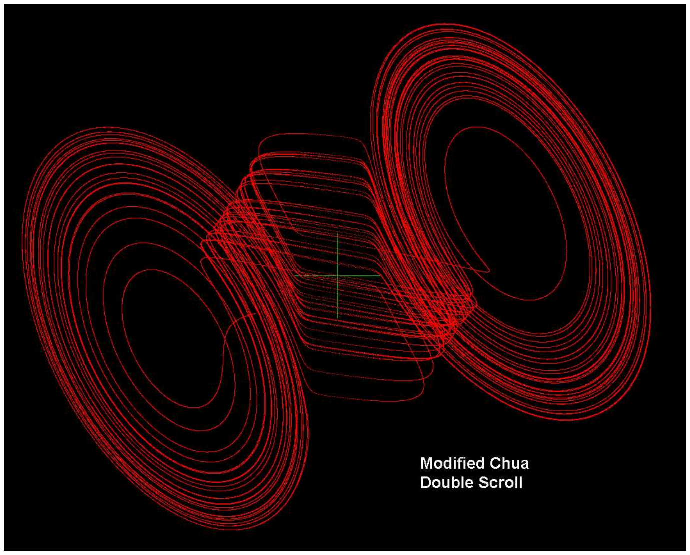

Figure 1: The Modified Chua Scroll

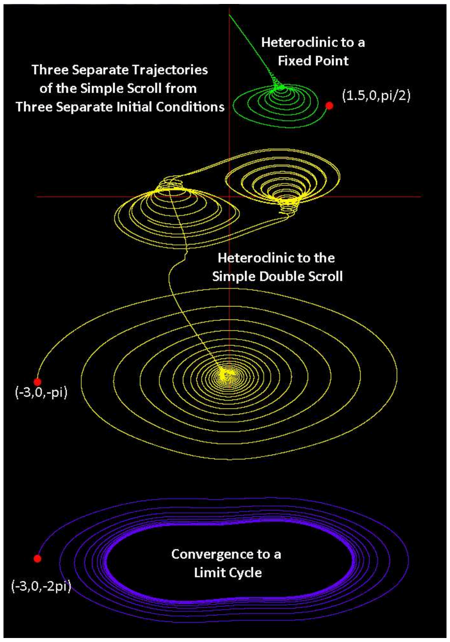

Figure 2: Three Trajectories of the Simple Scroll with Three Different Initial Conditions

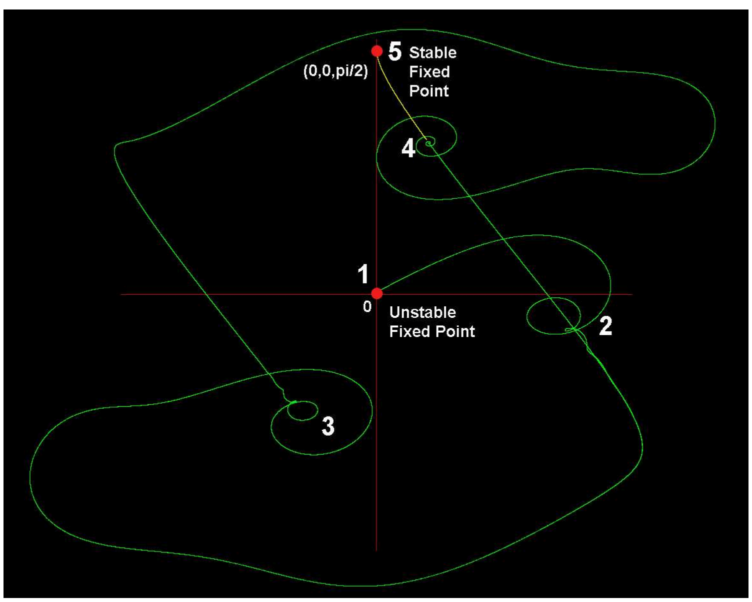

Figure 3: Complex Heteroclinic trajectory from the origin $(0,0,0)$ to $(0,0,\pi/2)$.

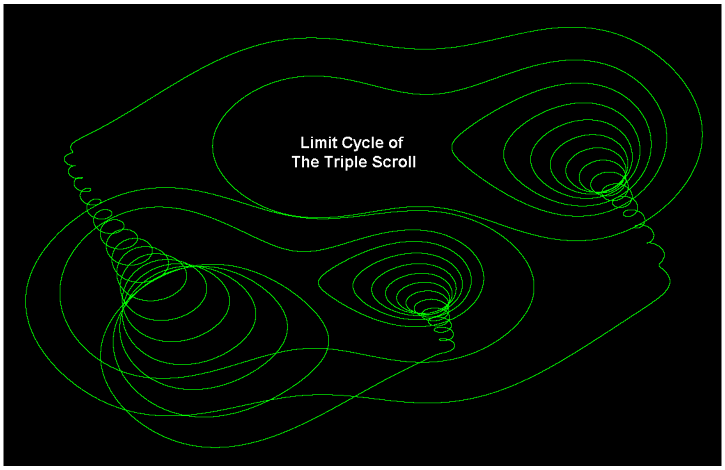

Figure 4: The Limit Cycle of the Triple Scroll

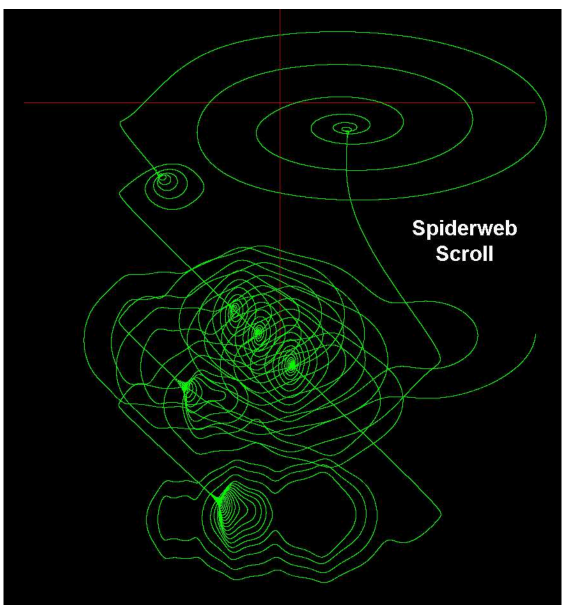

Figure 5: The Spiderweb Scroll

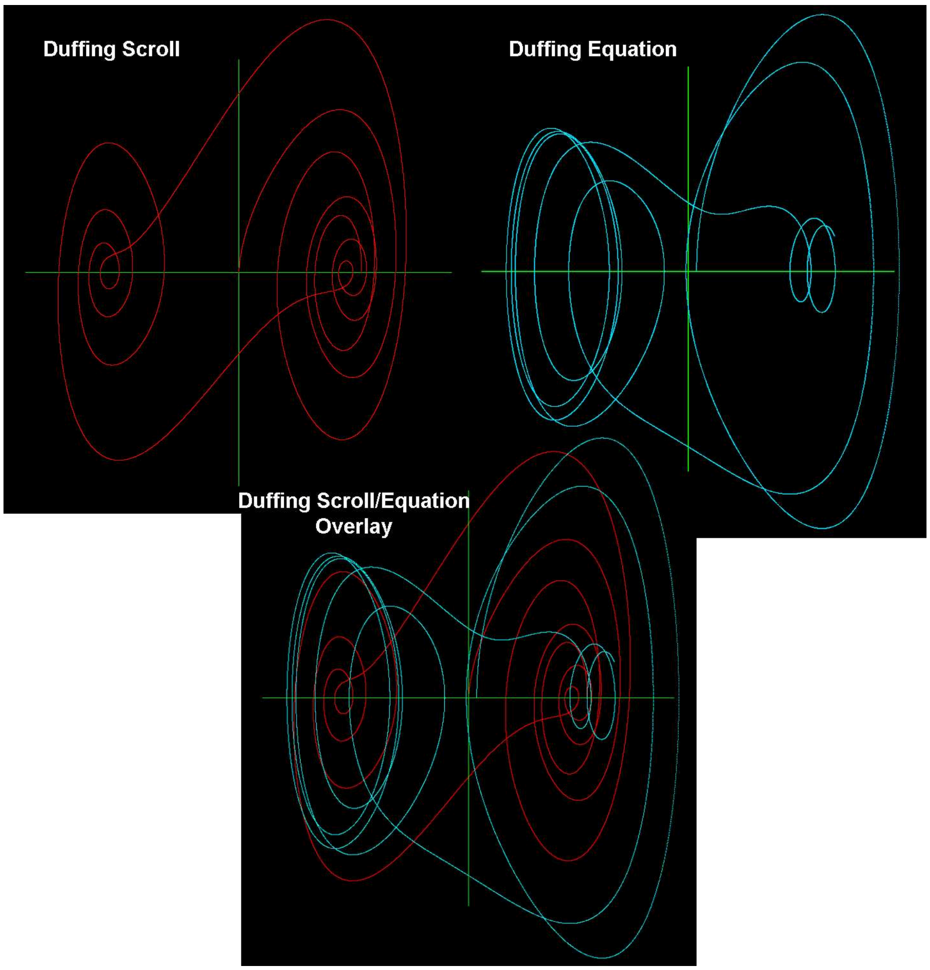

Figure 6: The Duffing Scroll